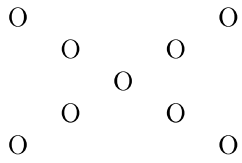
CMIMC 2021

Combinatorics & Computer Science Div. 2

1. We have a 9 by 9 chessboard with 9 kings (which can move to any of 8 adjacent squares) in the bottom row. What is the minimum number of moves, if two pieces cannot occupy the same square at the same time, to move all the kings into an X shape (a 5×5 region where there are 5 kings along each diagonal of the X, as shown below)?



2. Dilhan has objects of 3 types, A, B, and C, and 6 functions

$$f_{A,B}, f_{A,C}, f_{B,A}, f_{B,C}, f_{C,A}, f_{C,B}$$

where $f_{X,Y}$ takes in an object of type X and outputs an object of type Y. Dilhan wants to compose his 6 functions, without repeats, such that the resulting expression is well-typed, meaning an object can be taken in by the first function, and the resulting output can then be taken in by the second function, and so on. In how many orders can he compose his 6 functions, satisfying this constraint?

3. Adam has a box with 15 pool balls in it, numbered from 1 to 15, and picks out 5 of them. He then sorts them in increasing order, takes the four differences between each pair of adjacent balls, and finds exactly two of these differences are equal to 1. How many selections of 5 balls could he have drawn from the box?

4. Vijay has a stash of different size stones: in particular, he has 2021 types of stones, with sizes from 0 through 2020, and he has $2r + 1$ stones of size r.

   Vijay starts randomly (and without replacement) taking out stones from his stash and laying them out in a line. Vijay notices that the first stone of size 2020 comes before the first stone of size 2019, the first stone of size 2019 is before the first stone of size 2018, and so on. What is the probability of this happening?

   Express your answer in terms of only basic arithmetic operations (division, exponentiation, etc.) and the factorial function.

5. Bill Gates and Jeff Bezos are playing a game. Each turn, a coin is flipped, and if Bill and Jeff have $m, n > 0$ dollars, respectively, the winner of the coin toss will take $\min(m, n)$ from the loser. Given that Bill starts with 20 dollars and Jeff starts with 21 dollars, what is the probability that Bill ends up with all of the money?

6. Adam is playing Minesweeper on a 9×9 grid of squares, where exactly $\frac{1}{3}$ (or 27) of the squares are mines (generated uniformly at random over all such boards). Every time he clicks on a square, it is either a mine, in which case he loses, or it shows a number saying how many of the (up to eight) adjacent squares are mines.

   First, he clicks the square directly above the center square, which shows the number 4. Next, he clicks the square directly below the center square, which shows the number 1. What is the probability that the center square is a mine?

7. How many permutations of the string 0123456 are there such that no contiguous substrings of lengths $1 < \ell < 7$ have a sum of digits divisible by 7?

8. Suppose you have a 6 sided dice with 3 faces colored red, 2 faces colored blue, and 1 face colored green. You roll this dice 20 times and record the color that shows up on top. What is the expected value of the product of the number of red faces, blue faces, and green faces?