CMIMC 2021

Combinatorics & CS Div. 1 Solutions

1. Adam has a box with 15 pool balls in it, numbered from 1 to 15, and picks out 5 of them. He then sorts them in increasing order, takes the four differences between each pair of adjacent balls, and finds exactly two of these differences are equal to 1. How many selections of 5 balls could he have drawn from the box?

   Proposed by Adam Bertelli

   Answer: 990

   Solution: There are two cases:

   - If the two differences that are equal to 1 occur between disjoint pairs of elements (e.g. 1,2 and 4,5), then this is equivalent to selecting 3 objects out of 11, adding two dividers between pairs of objects to ensure that objects we do not want to be adjacent are not adjacent, and finally choosing two of our objects to represent a pair of adjacent balls instead of one ball. In total, this can occur in $\binom{11}{3}\binom{3}{2} = 465$ ways.
   - If instead the two differences that are equal to 1 have a shared element (e.g. 1,2 and 2,3), then this is again equivalent to selecting 3 objects out of 11, adding two dividers, and choosing one of our objects to represent 3 adjacent balls, which gives a total of $\binom{11}{3}\binom{3}{1} = 465$ ways.

   Thus in total there are $465 + 465 = \boxed{990}$ ways to select these 5 balls.

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2. Adam is playing Minesweeper on a 9×9 grid of squares, where exactly $\frac{1}{3}$ (or 27) of the squares are mines (generated uniformly at random over all such boards). Every time he clicks on a square, it is either a mine, in which case he loses, or it shows a number saying how many of the (up to eight) adjacent squares are mines.

   First, he clicks the square directly above the center square, which shows the number 4. Next, he clicks the square directly below the center square, which shows the number 1. What is the probability that the center square is a mine?

   Proposed by Adam Bertelli

   Answer: $\frac{88}{379}$

   Solution: Note that the squares touching both the 1 and the 4 form a rectangle of 3 squares. We will consider two separate cases on this rectangle:

   - If the central rectangle contains one mine, then there are $\binom{5}{3} = 10$ ways to place the mines around the 4, 3 ways to place the mine in the central rectangle, and $\binom{66}{23}$ ways to place the remaining 23 mines in squares that do not touch either the 1 or the 4. Note that in this case, there is a $\frac{1}{3}$ probability of the center square being a mine.
   - If the central rectangle contains no mines, then there are $\binom{5}{4} = 5$ ways to place the mines around the 4, 5 ways to place the mine around the 1, and $\binom{66}{22}$ ways to place all of the remaining mines. Note that in this case, there is no chance the center square is a mine.

   Thus our total probability for the center square being a mine is

$$\frac{\frac{1}{3} \cdot 3 \cdot 10 \cdot \binom{66}{23}}{3 \cdot 10 \cdot \binom{66}{23} + 5 \cdot 5 \cdot \binom{66}{22}} = \frac{2 \cdot \binom{66}{23}}{6 \cdot \binom{66}{23} + 5 \cdot \frac{23}{44} \cdot \binom{66}{23}} = \frac{2}{6 + \frac{115}{44}} = \boxed{\frac{88}{379}}.$$

Comment: This problem illustrates the more general fact that, for minesweeper boards with density < 0.5, individual configurations with less mines are relatively *more likely* to be correct, a useful trick often overlooked by minesweeper players.

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3. How many permutations of the string 0123456 are there such that no contiguous substrings of lengths $1 < \ell < 7$ have a sum of digits divisible by 7?

Proposed by Srinivasan Sathiamurthy

Answer: 420

Solution: Add 0 at the end, multiplying the total answer by 5 (we can insert it anywhere but the ends).

If we first require that no two adjacent digits sum to 7, there are 4 possible shapes of strings, where each letter denotes some pair from (1,6), (2,5), (3,4):

ABCABC
ABCACB
ABCBAC
ABACBC

The number of ways to place digits in each shape without restrictions is $2^3 \cdot 3! = 48$. The only additional restrictions are the tuples (1,2,4) and (3,5,6), which we can count using PIE. There are 4, 3, 3, and 2 places where such a contiguous subset could occur in each of the 4 shapes respectively, and there are 3! ways to rearrange each instance, so in total $3! \cdot 12 = 72$ strings contain (1,2,4) in some order, and by symmetry 72 contain (3,5,6). The only way to contain both is to have the first half be one, and the second half be the other, which only works in the first 3 shapes, and within these we can swap the two halves and rearrange one of the two halves, giving $2 \cdot 3! = 12$ degrees of freedom, thus the final answer is $4 \cdot 48 - 2 \cdot 72 + 3 \cdot 12 = 84$, which gives $\boxed{420}$ when multiplied by 5.

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4. Suppose you have a 6 sided dice with 3 faces colored red, 2 faces colored blue, and 1 face colored green. You roll this dice 20 times and record the color that shows up on top. What is the expected value of the product of the number of red faces, blue faces, and green faces?

Proposed by Daniel Li

Answer: 190

Solution: Consider the set S of ordered triples (i, j, k) such that the ith roll is red, the jth roll is green, and the kth roll is blue. It is clear that the number of such triples is rgb, so we just have to find the expected value of the size of this set.

For each of the $20 \cdot 19 \cdot 18$ possible such triples (i, j, k), it will be a part of S with probability $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6}$. By Linearity of Expectation

$$\mathbb{E}[|S|] = \mathbb{E}\left[\sum_{(i,j,k)\in S} 1\right] = \sum_{(i,j,k)} \mathbb{P}[(i,j,k) \in S] = (20 \cdot 19 \cdot 18) \cdot \frac{1}{36} = \boxed{190}$$

Alternate Solution (without Linearity of Expectation): Let the number of red faces be r, blue faces be b, and green faces be g. We will compute for n dice rolls. The expected value is the sum of possible products weighted by the probability of that outcome. Thus, we compute

$$\sum_{r+b+g=n} (rbg) \left(\left(\frac{1}{2}\right)^r \left(\frac{1}{3}\right)^b \left(\frac{1}{6}\right)^g \right) \binom{n}{r, g, b}$$

We expand the multinomial and factor constants to get

$$= \sum_{r+b+g=n} (rbg) \left(\left(\frac{1}{2}\right)^r \left(\frac{1}{3}\right)^b \left(\frac{1}{6}\right)^g \right) \frac{n!}{r!g!b!}$$

$$= \sum_{r+b+g=n} \left(\left(\frac{1}{2}\right)^r \left(\frac{1}{3}\right)^b \left(\frac{1}{6}\right)^g \right) \frac{(n-3)!}{(r-1)!\,(g-1)!\,(b-1)!} n\,(n-1)\,(n-2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) \sum_{r+b+g=n} \left(\left(\frac{1}{2}\right)^{r-1} \left(\frac{1}{3}\right)^{b-1} \left(\frac{1}{6}\right)^{g-1} \right) \frac{(n-3)!}{(r-1)!\,(g-1)!\,(b-1)!} n\,(n-1)\,(n-2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) n\,(n-1)\,(n-2) \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)^{n-3} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) n\,(n-1)\,(n-2)\,.$$

Thus, as we have 20 dice rolls, the desired expected value is $\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) \cdot 20 \cdot 19 \cdot 18 = \boxed{190}$.

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5. There are exactly 7 possible tetrominos (groups of 4 connected squares in a grid):

Daniel has a 2×20210 rectangle and wants to tile the interior with tetrominos without overlaps, pieces sticking out, or extra pieces left over. Note that you are allowed to rotate tetrominos but not reflect them. For how many multisets of tetrominos (ie. an ordered tuple of how many of each tile he has) is it possible to exactly tile his 2×20210 rectangle?

Proposed by Dilhan Salgado

Answer: $\binom{5055}{3} + \binom{5054}{3}$

Solution: First note that the T, S, and Z pieces cannot be used in a tiling of a $2 \times n$ rectangle. This is because if you placed it down, it partitions the rectangle into 2 pieces with an odd number of grid cells. These pieces cannot be tiled with tetrominos, because each one covers an even number of cells. Thus we can only use the other four pieces.

Now let's consider all of the simple 'blocks' the we can use to cover a $2 \times n$ rectangle exactly. (A block is simple if it cannot be split into two smaller blocks).
$n = 2 \to$ One O piece.
$n = 4 \to$ Two I pieces.
$n = 4k + 4 \to$ Two L pieces, $2k$ I pieces.
$n = 4k + 4 \to$ Two J pieces, $2k$ I pieces.
$n = 4k + 6 \to$ 1 L, 1 J, $2k + 1$ I pieces.
(You can find these by trying all possible things on the left, and seeing all the ways you can continue).

Let lowercase o denote the number of O pieces, and similar for the other 3 relevant pieces. We can see the following conditions on $o, i, l, j \geq 0$:
$4o + 4i + 4l + 4j = 2 \cdot 20210 \implies o + i + l + j = 10105$ (from the total number of cells covered).
$i \equiv l \equiv j \pmod 2$ (This congruence is satisfied for any block, so it's true overall).

These conditions are sufficient, because for any solution to these equations, we can construct a valid solution, even just using the blocks {1O,2I,2L,2J,1L+1J+1I} .

Now it just remains to count how many solutions (o, i, j, l) there are to the above equations.
We split into two cases: i, j, l odd and i, j, l even.

i, j, l odd:
Note that $o+i+j+l$ is odd, so o must be even.
Thus we can write: $o = 2o', i = 2i'+1, j = 2j'+1, l = 2l'+1 \implies (2o'+2i'+2j'+2l')+3 = 10105 \implies 2o'+2i'+2j'+2l' = 10102 \implies o'+i'+j'+l' = 5051$, where $o', i', j', l' \geq 0$.
By stars and bars, there are $\binom{5054}{3}$ solutions to this.

i, j, l even:
Note that $o+i+j+l$ is odd, so o must be odd.
Thus we can write: $o = 2o'+1, i = 2i', j = 2j', l = 2l' \implies (2o'+2i'+2j'+2l')+1 = 10105 \implies 2o'+2i'+2j'+2l' = 10104 \implies o'+i'+j'+l' = 5052$, where $o', i', j', l' \geq 0$.
By stars and bars, there are $\binom{5055}{3}$ solutions to this.

Thus the total number of (o, i, j, l) that work is $\boxed{\binom{5055}{3}+\binom{5054}{3}}$.

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6. Alice and Bob each flip 20 fair coins. Given that Alice flipped at least as many heads as Bob, what is the expected number of heads that Alice flipped?

Proposed by Adam Bertelli

Answer: $\dfrac{20 \cdot \left(2^{39}+\binom{40}{20}\right)}{2^{40}+\binom{40}{20}}$

Solution: Consider a 21×21 matrix A, where $A_{i,j} = \binom{20}{i}\binom{20}{j}$ is the number of ways for Alice to flip i heads, and Bob to flip j heads. Now, consider the matrix B satisfying $B_{i,j} = iA_{i,j}$, where each term is weighted by the number of heads Alice flipped. The answer we are looking for is the sum of the lower half-triangle of B divided by the sum of the lower half-triangle of A.

For A, computing this sum is easy, as A is diagonally symmetric, so the desired sum is simply half of the total sum of A, plus half of the sum of the diagonal of A. The total sum of A is $\left(\sum_{i=0}^{20}\binom{20}{i}\right)^2 = 2^{40}$, and the sum of the diagonal is $\sum_{i=0}^{20}\binom{20}{i}^2 = \binom{40}{20}$, so the total value is $\frac{1}{2}\left(2^{40}+\binom{40}{20}\right)$.

For B, we will first rewrite each term with $i > 0$ as $B_{i,j} = 20\binom{19}{i-1}\binom{20}{j}$ (note that $B_{0,j} = 0$ so we can ignore these terms). This tells us that the submatrix C given by taking the 19 rows where $i > 0$, and all 20 columns, is both horizontally and vertically symmetric, thus it's lower half triangle sum is simply equal to half of it's total sum, or $\frac{1}{2} \cdot 20\left(\sum_{i=1}^{20}\binom{19}{i-1}\right)\left(\sum_{j=0}^{20}\binom{20}{j}\right) = \frac{1}{2} \cdot 20 \cdot 2^{39}$.

However, note that the lower half triangle sum of C only contains all cells s.t. $i > j$, thus we must add in the entire diagonal. The diagonal sum is equal to $\sum_{i=1}^{20} 20\binom{19}{i-1}\binom{20}{20-i} = 20 \cdot \binom{39}{19} = 10 \cdot \binom{40}{20}$ by Vandermonde's, so in total our desired sum for B is $\frac{1}{2} \cdot 20\left(2^{39}+\binom{40}{20}\right)$.

Finally, taking the quotient of these two values gives us our desired answer, which evaluates to roughly 11.11.

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7. How many non-decreasing tuples of integers $(a_1, a_2, \ldots, a_{16})$ are there such that $0 \leq a_i \leq 16$ for all i, and the sum of all a_i is even?

Proposed by Nancy Kuang

Answer: $\frac{1}{2}\left(\binom{32}{16}+\binom{16}{8}\right)$

Solution: Let $k = 8$. Every tuple $(a_1, \ldots, a_{2k})$ can be bijected to an up-right lattice path from $(0, 0)$ to $(2k, 2k)$, and the sum of all a_i is equal to the area of the region bounded by the path and the lines $y = 0$ and $x = 2k$. Define an odd point to be a point with two odd coordinates. We have two cases:

- The path doesn't pass through any odd points. Then the area is even, and there are $\binom{2k}{k}$ paths satisfying this.
- The path passes through at least one odd point. Exactly half of these will give an even area: consider the first odd point that the path passes through. The last two steps taken to get to that point are either (a) $(0, +1)$ followed by $(+1, 0)$, or (b) $(+1, 0)$ followed by $(0, +1)$. Note that reversing the last two steps changes the parity of the area. Then the number of paths passing through an odd point resulting in an even area is $\dfrac{\binom{4k}{2k} - \binom{2k}{k}}{2}$.

Adding the two, we get $\frac{\binom{4k}{2k}+\binom{2k}{k}}{2}$, and plugging in $k = 8$ gives $\boxed{\dfrac{\binom{32}{16} + \binom{16}{8}}{2}}$.

8. An *augmentation* on a graph G is defined as doing the following:

- Take some set D of vertices in G, and duplicate each vertex $v_i \in D$ to create a new vertex v_i'.

- If there's an edge between a pair of vertices $v_i, v_j \in D$, create an edge between vertices v_i' and v_j'. If there's an edge between a pair of vertices $v_i \in D$, $v_j \notin D$, you can choose to create an edge between v_i' and v_j but do not have to.

A graph is called *reachable* from G if it can be created through some sequence of augmentations on G. Some graph H has n vertices and satisfies that both H and the complement of H are reachable from a complete graph of 2021 vertices. If the maximum and minimum values of n are M and m, find $M + m$.

Proposed by Oliver Hayman

Answer: $2021^2 + 4041$

Solution: The maximum is 2021^2, and the minimum is $2021 + 2020$.

Notice that the chromatic number of any graph reachable from G is the same as the chromatic number of G. To show this, let the chromatic number of G be a, and the chromatic number of some graph G' that is reached by performing an *augmentation* on G be a'. If every vertex v_i' in the *augmentation* is colored the same color as v_i this creates a valid coloring of G' with D colors, so $a \geq a'$. Additionally, since G is a subgraph of G' we have $a' \geq a$, so $a = a'$.

This means H and the complement of H have chromatic number 2021. For some coloring of H, consider sets $S_1, S_2, \cdots, S_{2021}$ of nodes of H such that S_i contains nodes of the same color. All nodes in S_i are nonadjacent, so in the complement of H the nodes in S_i will form a $K_{|S_i|}$. Since the chromatic number of the complement of H is 2021, we have $|S_i| \leq 2021$ for all i, so $M \leq 2021^2$. This is achieved when H is 2021 copies of a K_{2021}, as H can be achieved by choosing the original K_{2021} in G and duplicating it for 2020 *augmentations*, where no additional edges are added. Additionally, we can achieve the complement of H first by coloring G using 2021 colors. Whenever we duplicate a node v_i to create v_i' we color v_i' the same color as v_i. Then if we duplicate the original K_{2021} in G for 2020 *augmentations* and add edges between each newly created node and each node of a different color, we will create the complement of H. This gives $M = 2021^2$.

Note that since the complement of H must contain a K_{2021}, there must be a set of 2021 nodes in H where no two are adjacent. H also contains a K_{2021}, and since we can have at most one node of the K_{2021} in the independent set we will require an additional 2020 nodes, giving $m \geq 4041$. If H is the graph consisting of a K_{2021} and 2020 isolated nodes, H can be achieved by duplicating a single node 2020 times and choosing to add no additional edges. Additionally, the complement of H can be achieved by duplicating the same node of G 2020 times and each time adding edges from the duplicated node to every other node in the original K_{2021}. This gives $m = 4041$.

We have $M + m = 4041 + 2021^2 = \boxed{4088482}$.