CMIMC 2021

Combinatorics & Computer Science Div. 1

1. Adam has a box with 15 pool balls in it, numbered from 1 to 15, and picks out 5 of them. He then sorts them in increasing order, takes the four differences between each pair of adjacent balls, and finds exactly two of these differences are equal to 1. How many selections of 5 balls could he have drawn from the box?

2. Adam is playing Minesweeper on a 9×9 grid of squares, where exactly $\frac{1}{3}$ (or 27) of the squares are mines (generated uniformly at random over all such boards). Every time he clicks on a square, it is either a mine, in which case he loses, or it shows a number saying how many of the (up to eight) adjacent squares are mines.

   First, he clicks the square directly above the center square, which shows the number 4. Next, he clicks the square directly below the center square, which shows the number 1. What is the probability that the center square is a mine?

3. How many permutations of the string 0123456 are there such that no contiguous substrings of lengths $1 < \ell < 7$ have a sum of digits divisible by 7?

4. Suppose you have a 6 sided dice with 3 faces colored red, 2 faces colored blue, and 1 face colored green. You roll this dice 20 times and record the color that shows up on top. What is the expected value of the product of the number of red faces, blue faces, and green faces?

5. There are exactly 7 possible tetrominos (groups of 4 connected squares in a grid):

   Daniel has a 2×20210 rectangle and wants to tile the interior with tetrominos without overlaps, pieces sticking out, or extra pieces left over. Note that you are allowed to rotate tetrominos but not reflect them. For how many multisets of tetrominos (ie. an ordered tuple of how many of each tile he has) is it possible to exactly tile his 2×20210 rectangle?

6. Alice and Bob each flip 20 fair coins. Given that Alice flipped at least as many heads as Bob, what is the expected number of heads that Alice flipped?

7. How many non-decreasing tuples of integers $(a_1, a_2, \ldots, a_{16})$ are there such that $0 \le a_i \le 16$ for all i, and the sum of all a_i is even?

8. An *augmentation* on a graph G is defined as doing the following:

   - Take some set D of vertices in G, and duplicate each vertex $v_i \in D$ to create a new vertex v_i'.

   - If there's an edge between a pair of vertices $v_i, v_j \in D$, create an edge between vertices v_i' and v_j'. If there's an edge between a pair of vertices $v_i \in D$, $v_j \notin D$, you can choose to create an edge between v_i' and v_j but do not have to.

   A graph is called *reachable* from G if it can be created through some sequence of augmentations on G. Some graph H has n vertices and satisfies that both H and the complement of H are reachable from a complete graph of 2021 vertices. If the maximum and minimum values of n are M and m, find $M + m$.