Algebra & Number Theory Div. 2 Solutions

1. Find the unique 3 digit number $N = \underline{A}\ \underline{B}\ \underline{C}$, whose digits (A, B, C) are all nonzero, with the property that the product $P = \underline{A}\ \underline{B}\ \underline{C} \times \underline{A}\ \underline{B} \times \underline{A}$ is divisible by 1000.

   Proposed by Kyle Lee

   Answer: 875

   Solution: First, note that $1000 = 2^3 \cdot 5^3$, so it suffices to distribute 3 factors of 2 and 3 factors of 5 into 3 blocks of positive integers. If any of them receives at least 1 factor of 2 and 1 factor of 5, it will be divisible by 10 and hence have a units digit of 0, contradiction. Therefore, at least one of the numbers will be divisible by 2^2 and another will be divisible by 5^2.

   With this in mind, we casework on how the factors of 5 distribute. If $A = 5$, then we require $\underline{5}\ \underline{B}\ \underline{C}$ to contain a factor of 5^2 since $\underline{5}\ \underline{B}$ cannot be divisible by 5^2 (this forces $B = 0$). But then this implies $\underline{5}\ \underline{B}$ is divisible by 2^3, so we must have $B = 6$. There is an obvious contradiction then since $\underline{5}\ \underline{6}\ \underline{C}$ cannot be divisible by 5^2.

   Hence, suppose $A \neq 5$. If both $\overline{A}\ \overline{B}\ \overline{C}$ and $\overline{A}\ \overline{B}$ contain a factor of 5, then we must have $A = 8$. Clearly, we must have $\overline{8}\ \overline{B}\ \overline{C}$ contain factor of 5^2 and $\overline{8}\ \overline{B}$ contain a factor of 5 since the latter cannot possibly contain a factor of 5^2. But this forces $B = 5$ and evidently $\overline{8}\ \overline{5}\ \overline{C}$ cannot contain a factor of 5^2 unless $C = 0$, contradiction.

   Thus, we have boiled down to the final case where $A \neq 5$ and $\overline{A}\ \overline{B}\ \overline{C}$ contains a factor of 5^3 since $\overline{A}\ \overline{B}$ cannot contain a factor of 5^3. From here, it is easy to check that $\boxed{875}$ is the only number that works.

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2. Suppose a, b are positive real numbers such that $a + a^2 = 1$ and $b^2 + b^4 = 1$. Compute $a^2 + b^2$.

   Proposed by Thomas Lam

   Answer: 1

   Solution: Consider the positive real solution to the quadratic equation $x + x^2 = 1$. Then $x + x^2 = a + a^2$ so $x = a$. Likewise, since $b^2 + b^4 = 1$ we have that $b^2 = x$.
   Then $a^2 + b^2 = x^2 + x = \boxed{1}$.

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3. How many multiples of 12 divide 12! and have exactly 12 divisors?

   Proposed by Adam Bertelli

   Answer: 6

   Solution: The prime factorization of 12! is $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, and 12 can factor into a product of integers > 1 in the following ways:
   $$12 = 3 \cdot 2 \cdot 2 = 6 \cdot 2 = 4 \cdot 3.$$
   For a divisor d of 12! with 12 divisors, the set of exponents in its prime factorization must then be one of $\{11\}, \{2, 1, 1\}, \{5, 1\}, \{3, 2\}$. For d to be a multiple of 12, the exponent on 2 must be at least 2 and the exponent on 3 must be at least 1. The corresponding possibilities for d are
   $$2^2 \cdot 3 \cdot 5, \quad 2^2 \cdot 3 \cdot 7, \quad 2^2 \cdot 3 \cdot 11, \quad 2^5 \cdot 3, \quad 2^3 \cdot 3^2, \quad 2^2 \cdot 3^3$$
   for a total of $\boxed{6}$.

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4. What is the 101st smallest integer which can represented in the form $3^a + 3^b + 3^c$, where a, b, and c are integers?

Proposed by Dilhan Salgado

Answer: 2431

Solution: Call a number *valid* if it is of the desired form, and let $0 < a_1 < a_2 < \cdots$ be the sequence of valid integers. There are exactly three possible forms of the ternary expansion for a valid n, namely:

- a single 1 and all other digits 0 (occurring when $a = b = c$),
- a single 1, a single 2, and all other digits 0 (occuring when $a = b \neq c$),
- three 1's and all other digits 0 (when a, b, c are pairwise distinct).

Let $f(k)$ denote the number of valid integers using at most k ternary digits. Clearly $f(k) = k + 2\binom{k}{2} + \binom{k}{3}$. Since $f(7) = 84 < 101 < 120 = f(8)$, we conclude that $3^7 < a_{101} < 3^8$.

Now let $3^7 < b_1 < b_2 < \cdots$ be the sequence of valid numbers $> 3^7$. Since 3^7 is itself valid, $a_{101} = b_{16}$. Let $c_i = b_i - 3^7$. A number $0 < d < 3^7$ is of the form c_i for some i if and only if its ternary digits sum to 2. With at most k digits, then, there are $\binom{k+1}{2}$ numbers $< 3^7$ of the form c_i. With $k = 5$, we then see that there are 15 numbers $< 3^7$ of the form c_i using ≤ 5 digits. It follows that $c_{16} = 100001_3$, so $a_{101} = 10100001_3$, which is $\boxed{2431}$.

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5. Suppose there are 160 pigeons and n holes. The 1st pigeon flies to the 1st hole, the 2nd pigeon flies to the 4th hole, and so on, such that the ith pigeon flies to the $(i^2 \bmod n)$th hole, where $k \bmod n$ is the remainder when k is divided by n. What is minimum n such that there is at most one pigeon per hole?

Proposed by Christina Yao

Answer: 326

Solution: Note that $a^2 \equiv b^2 \mod n$ iff $(a + b)(a - b) \equiv 0 \mod n$. Equivalently, n cannot be a factor of $(a + b)(a - b)$ for all distinct $a, b \leq 160$. This leaves two possibilities:

- $n = p$ for $p > 320$.
- $n = 2p$ for $p > 160$, since $a + b = p \implies a - b$ is odd, so the product cannot be divisible by $2p$.

By checking numbers above 320, we can see that the first number satisfying one of these conditions is $n = 2 \cdot 163 = \boxed{326}$.

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6. Let a and b be complex numbers such that $(a+1)(b+1) = 2$ and $(a^2+1)(b^2+1) = 32$. Compute the sum of all possible values of $(a^4 + 1)(b^4 + 1)$.

Proposed by Kyle Lee

Answer: 1160

Solution: Rewrite the first equation as $a + b = 1 - ab$, so that

$$\begin{aligned} a^2 + b^2 &= (a + b)^2 - 2ab \\ &= (1 - ab)^2 - 2ab \\ &= 1 - 4ab + (ab)^2. \end{aligned}$$

Therefore, $1 - 4ab + 2(ab)^2 = 31$, so $ab = -3$ or 5. Now,

$$\begin{aligned}(a^4+1)(b^4+1) &= (ab)^4 + a^4 + b^4 + 1\\ &= (ab)^4 + (a^2+b^2)^2 - 2(ab)^2 + 1\\ &= (ab)^4 + (31-(ab)^2)^2 - 2(ab)^2 + 1.\end{aligned}$$

If $ab = -3$, the expression evaluates to 548. However, if $ab = 5$, the expression evaluates to 612, so the answer is $548 + 612 = \boxed{1160}$.

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7. For each positive integer n, let $\sigma(n)$ denote the sum of the positive integer divisors of n. How many positive integers $n \leq 2021$ satisfy

$$\sigma(3n) \geq \sigma(n) + \sigma(2n)?$$

Proposed by Kyle Lee

Answer: 1481

Solution: Call a positive integer $n \leq 2021$ bad if $\sigma(3n) < \sigma(n) + \sigma(2n)$. We will compute the number of bad $n \leq 2021$ and subtract this from 2021.

Suppose $2^a \times 3^b \times \cdots$ is the prime factorization of n. It is well-known that the sum of the factors of n is

$$\begin{aligned}&(1+2+2^2+\cdots+2^a)(1+3+3^2+\cdots+3^b)\cdots\\ &= \left(\frac{2^{a+1}-1}{1}\right)\left(\frac{3^{b+1}-1}{2}\right)\cdots.\end{aligned}$$

Then since we only care about factors of 2 or 3, we have

$$\left(\frac{2^{a+1}-1}{1}\right)\left(\frac{3^{b+2}-1}{2}\right) < \left(\frac{2^{a+1}-1}{1}\right)\left(\frac{3^{b+1}-1}{2}\right) + \left(\frac{2^{a+2}-1}{1}\right)\left(\frac{3^{b+1}-1}{2}\right).$$

If we let $2^a = x$ and $3^b = y$, the equation can be rewritten as

$$(2x-1)(9y-1) < (2x-1)(3y-1) + (4x-1)(3y-1),$$

which easily simplifies to $4x - 3y < 1$, or equivalently $2^{a+2} - 3^{b+1} < 1$.

Now, $b > 0$ since otherwise $2^{a+2} - 3 < 1$, contradiction. There are $\left\lfloor \frac{2021}{3} \right\rfloor = 673$ multiples of 3. However, not all of them work. Let us casework on the value of $1 \leq b \leq 6$ (since $3^6 < 2021 < 3^7$).

The pair $(a, b) = (2, 1)$ fails to satisfy $2^{a+2} - 3^{b+1} < 1$, but the pair $(a, b) = (2, 2)$ works. So we need to subtract off $\left\lfloor \frac{2021}{12} \right\rfloor - \left\lfloor \frac{2021}{36} \right\rfloor = 112$ values of n.

Similarly, the pair $(a, b) = (3, 2)$ fails, but $(a, b) = (3, 3)$ works. So we need to subtract off $\left\lfloor \frac{2021}{72} \right\rfloor - \left\lfloor \frac{2021}{216} \right\rfloor = 19$ values of n. Note that this does not overlap with the first case since 72 is a multiple of 36.

Similarly, the pair $(a, b) = (5, 3)$ fails, but $(a, b) = (5, 4)$ works. So we need to subtract off $\left\lfloor \frac{2021}{864} \right\rfloor - \left\lfloor \frac{2021}{2592} \right\rfloor = 2$ values of n. Note that this does not overlap with either the first or second case since 864 is a multiple of both 36 and 216.

Now, for $b > 3$, there are other pairs that fail, but it will not matter since $2^6 \cdot 3^4 > 2021$.

Lastly, there are $673 - (112 + 19 + 2) = 540$ bad $n \leq 2021$, so the answer is $2021 - 540 = \boxed{1481}$.

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8. Let $f(x) = \frac{x^2}{8}$. Starting at the point $(7, 3)$, what is the length of the shortest path that touches the graph of f, and then the x-axis?

Proposed by Sam Delatore

Answer: $5\sqrt{2}-2$

Solution: The key to this problem is that, for any point on a parabola, it is equidistant from the focus and the directrix. From this, it's not hard to see that the shortest path from $(7,3)$ to the parabola to the directrix has the same length as the segment connecting $(7,3)$ to the focus. Here, the focus of this parabola is $(0,2)$, and the directrix is the line $y=-2$, which makes that distance equal to $5\sqrt{2}$. Since the desired length is two less than that of the shortest path from $(7,3)$ to the parabola to the line $y=-2$, we get an answer of $\boxed{5\sqrt{2}-2}$.