Algebra & Number Theory Div. 2

1. Find the unique 3 digit number $N = \underline{A}\ \underline{B}\ \underline{C}$, whose digits (A, B, C) are all nonzero, with the property that the product $P = \underline{A}\ \underline{B}\ \underline{C} \times \underline{A}\ \underline{B} \times \underline{A}$ is divisible by 1000.

2. Suppose a, b are positive real numbers such that $a + a^2 = 1$ and $b^2 + b^4 = 1$. Compute $a^2 + b^2$.

3. How many multiples of 12 divide 12! and have exactly 12 divisors?

4. What is the 101st smallest integer which can represented in the form $3^a + 3^b + 3^c$, where a, b, and c are integers?

5. Suppose there are 160 pigeons and n holes. The 1st pigeon flies to the 1st hole, the 2nd pigeon flies to the 4th hole, and so on, such that the ith pigeon flies to the $(i^2 \bmod n)$th hole, where $k \bmod n$ is the remainder when k is divided by n. What is minimum n such that there is at most one pigeon per hole?

6. Let a and b be complex numbers such that $(a + 1)(b + 1) = 2$ and $(a^2 + 1)(b^2 + 1) = 32$. Compute the sum of all possible values of $(a^4 + 1)(b^4 + 1)$.

7. For each positive integer n, let $\sigma(n)$ denote the sum of the positive integer divisors of n. How many positive integers $n \leq 2021$ satisfy
$$\sigma(3n) \geq \sigma(n) + \sigma(2n)?$$

8. Let $f(x) = \frac{x^2}{8}$. Starting at the point $(7, 3)$, what is the length of the shortest path that touches the graph of f, and then the x-axis?