CMIMC 2021

Algebra & Number Theory Div. 1 Solutions

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1. How many multiples of 12 divide 12! and have exactly 12 divisors?

   Proposed by Adam Bertelli

   Answer: 6

   Solution: The prime factorization of 12! is $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, and 12 can factor into a product of integers > 1 in the following ways:
   $$12 = 3 \cdot 2 \cdot 2 = 6 \cdot 2 = 4 \cdot 3.$$
   For a divisor d of 12! with 12 divisors, the set of exponents in its prime factorization must then be one of $\{11\}, \{2, 1, 1\}, \{5, 1\}, \{3, 2\}$. For d to be a multiple of 12, the exponent on 2 must be at least 2 and the exponent on 3 must be at least 1. The corresponding possibilities for d are
   $$2^2 \cdot 3 \cdot 5, \quad 2^2 \cdot 3 \cdot 7, \quad 2^2 \cdot 3 \cdot 11, \quad 2^5 \cdot 3, \quad 2^3 \cdot 3^2, \quad 2^2 \cdot 3^3$$
   for a total of $\boxed{6}$.

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2. Suppose there are 160 pigeons and n holes. The 1st pigeon flies to the 1st hole, the 2nd pigeon flies to the 4th hole, and so on, such that the ith pigeon flies to the $(i^2 \bmod n)$th hole, where $k \bmod n$ is the remainder when k is divided by n. What is minimum n such that there is at most one pigeon per hole?

   Proposed by Christina Yao

   Answer: 326

   Solution: Note that $a^2 \equiv b^2 \mod n$ iff $(a + b)(a - b) \equiv 0 \mod n$. Equivalently, n cannot be a factor of $(a + b)(a - b)$ for all distinct $a, b \leq 160$. This leaves two possibilities:

   - $n = p$ for $p > 320$.
   - $n = 2p$ for $p > 160$, since $a + b = p \implies a - b$ is odd, so the product cannot be divisible by $2p$.

   By checking numbers above 320, we can see that the first number satisfying one of these conditions is $n = 2 \cdot 163 = \boxed{326}$.

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3. Let a and b be complex numbers such that $(a+1)(b+1) = 2$ and $(a^2+1)(b^2+1) = 32$. Compute the sum of all possible values of $(a^4 + 1)(b^4 + 1)$.

   Proposed by Kyle Lee

   Answer: 1160

   Solution: Rewrite the first equation as $a + b = 1 - ab$, so that
   $$\begin{aligned} a^2 + b^2 &= (a + b)^2 - 2ab \\ &= (1 - ab)^2 - 2ab \\ &= 1 - 4ab + (ab)^2. \end{aligned}$$
   Therefore, $1 - 4ab + 2(ab)^2 = 31$, so $ab = -3$ or 5. Now,
   $$\begin{aligned} (a^4 + 1)(b^4 + 1) &= (ab)^4 + a^4 + b^4 + 1 \\ &= (ab)^4 + (a^2 + b^2)^2 - 2(ab)^2 + 1 \\ &= (ab)^4 + (31 - (ab)^2)^2 - 2(ab)^2 + 1. \end{aligned}$$

If $ab = -3$, the expression evaluates to 548. However, if $ab = 5$, the expression evaluates to 612, so the answer is $548 + 612 = \boxed{1160}$.

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4. Let $f(x) = \frac{x^2}{8}$. Starting at the point $(7, 3)$, what is the length of the shortest path that touches the graph of f, and then the x-axis?

Proposed by Sam Delatore

Answer: $5\sqrt{2} - 2$

Solution: The key to this problem is that, for any point on a parabola, it is equidistant from the focus and the directrix. From this, it's not hard to see that the shortest path from $(7, 3)$ to the parabola to the directrix has the same length as the segment connecting $(7, 3)$ to the focus. Here, the focus of this parabola is $(0, 2)$, and the directrix is the line $y = -2$, which makes that distance equal to $5\sqrt{2}$. Since the desired length is two less than that of the shortest path from $(7, 3)$ to the parabola to the line $y = -2$, we get an answer of $\boxed{5\sqrt{2} - 2}$.

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5. Suppose f is a degree 42 polynomial such that for all integers $0 \le i \le 42$,

$$f(i) + f(43 + i) + f(2 \cdot 43 + i) + \cdots + f(46 \cdot 43 + i) = (-2)^i$$

Find $f(2021) - f(0)$.

Proposed by Adam Bertelli

Answer: $3^{43} - 2^{43} - 1$

Solution: Let $g(i)$ denote the LHS. We want to find $g(43) - g(0)$. Define $h(i) = \sum_{k=0}^{42} \binom{i}{k}(-3)^k$, which is a degree 42 polynomial. By the binomial theorem, for $0 \le i \le 42$, we have $h(i) = (-2)^i$. Since g and h agree at these 43 values, we have $g = h$. We compute

$$h(43) = \sum_{k=0}^{42} \binom{43}{k}(-3)^k = (-2)^{43} - (-3)^{43}$$

and $h(0) = 1$, so $g(43) - g(0) = h(43) - h(0) = \boxed{3^{43} - 2^{43} - 1}$.

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6. Find the remainder when

$$\left\lfloor \frac{149^{151} + 151^{149}}{22499} \right\rfloor$$

is divided by 10^4.

Proposed by Vijay Srinivasan

Answer: 7800

Solution: Let $p = 149, q = 151$, which are both prime. Observe that $22499 = 22500 - 1 = pq$. We have $p^q \equiv p \mod q$ and $q^p \equiv q \mod p$, so $p^q + q^p \equiv p + q \mod pq$, and thus

$$N = \frac{p^q + q^p - p - q}{pq}.$$

With $k = 150$, we have $(k-1)^{k+1} \equiv k(k+1) - 1 \mod k^2$ and $(k+1)^{k-1} \equiv k(k-1) + 1 \mod k^2$, so adding these gives

$$p^q + q^p = (k-1)^{k+1} + (k+1)^{k-1} \equiv 0 \mod k^2.$$

It follows that $p^q + q^p \equiv 0 \mod 5^4$, so $N \equiv p + q \equiv 300 \mod 5^4$.

Since, modulo 16, everything has order at most 4, we see that $p^q + q^p - p - q \equiv p^3 - p \mod 16$. Since $p \equiv 5 \mod 16$, we get that this expression is $8 \mod 16$. Since pq is odd, $N \equiv 8 \mod 16$ as well.

So for some integer m, we have $8 \equiv 5^4 m + 300 \equiv m + 12 \mod 16$, so $m \equiv -4 \mod 16$. So the answer is $-4 \cdot 5^4 + 300 \equiv -2200 \mod 10^4$, for a final answer of $10^4 - 2200 = \boxed{7800}$.

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7. As a gift, Dilhan was given the number $n = 1^1 \cdot 2^2 \cdots 2021^{2021}$, and each day, he has been dividing n by $2021!$ exactly once. One day, when he did this, he discovered that, for the first time, n was no longer an integer, but instead a reduced fraction of the form $\frac{a}{b}$. What is the sum of all distinct prime factors of b?

Proposed by Adam Bertelli

Answer: 354

Solution: First let us only consider primes $p > \sqrt{2021}$. Note that

$$\nu_p(n) = p + 2p + \cdots + d_p p = p d_p \frac{d_p + 1}{2}$$

where we define d_k as the largest integer such that $d_k k < 2021$, while

$$\nu_p(2021!) = 1 + 1 + \cdots + 1 = d_p$$

thus $\frac{\nu_p(n)}{\nu_p(2021!)} = \frac{p(d_p+1)}{2}$, or half of the least multiple of p larger than 2021. Such a number is at least 2022, and since $337 > \sqrt{2021}$ is a prime dividing 2022, it follows that 1011 is an attainable lower bound, meaning that after 1012 divisions, n is no longer an integer. It remains to find all p such that $\frac{\nu_p(n)}{\nu_p(2021!)} < 1012$.

We now extend our consideration to all primes p. For any given p, the final value of $\frac{\nu_p(n)}{\nu_p(2021!)}$ will be a weighted average of the summations given by taking one factor of p from each multiple of p, one factor of p from each multiple of p^2, and so on, i.e.

$$\frac{\nu_p(n)}{\nu_p(2021!)} = \frac{p d_p \frac{d_p+1}{2} + p^2 d_{p^2} \frac{d_{p^2}+1}{2} + p^3 d_{p^3} \frac{d_{p^3}+1}{2} + \cdots}{d_p + d_{p^2} + d_{p^3} + \cdots}$$

Clearly $p^k(d_{p^k} + 1) \geq p(d_p + 1)$, as every multiple of p^k is a multiple of p, thus in order for a prime to possibly have $\frac{\nu_p(n)}{\nu_p(2021!)} < 1012$, we must have $p | 2022$ or $p | 2023$, giving $p = 2, 3, 7, 17, 337$ as our possible values. We can rule out $2, 3, 7$ fairly quickly by computing prefix sums for our numerator and denominator until the weighted average exceeds 1012, and for 17, note that $17^2 | 2023$ and $17^3 > 2021$, hence $\frac{\nu_{17}(n)}{\nu_{17}(2021!)}$ is exactly $\frac{2023}{2}$, giving our final answer as $17 + 337 = \boxed{354}$.

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8. There are integers v, w, x, y, z and real numbers $0 \leq \theta < \theta' \leq \pi$ such that

$$\cos 3\theta = \cos 3\theta' = v^{-1}, \qquad w + x \cos\theta + y \cos 2\theta = z \cos\theta'.$$

Given that $z \neq 0$ and v is positive, find the sum of the 4 smallest possible values of v.

Proposed by Vijay Srinivasan

Answer: 36

Solution: Let $\alpha_1 = 2\cos\theta, \alpha_2 = 2\cos\theta'$; the constraint $\theta, \theta' \in [0, \pi]$ ensures that $\alpha_1 \neq \alpha_2$. By triple-angle formulas, α_1 and α_2 are roots to the equation $f(x) = x^3 - 3x - \frac{2}{v} = 0$. Let α_3 be the third root of f.

The condition with w, x, y, z ensures that $\alpha_2 \in F := \mathbb{Q}(\alpha_1)$, where $\mathbb{Q}(\alpha_1)$ denotes the set of rational linear combinations of $1, \alpha_1, \alpha_1^2$. This implies $F = \mathbb{Q}(\alpha_2) = \mathbb{Q}(\alpha_3)$. We compute $f(x) = (x - \alpha_1)g(x)$ where

$$g(x) = x^2 + \alpha_1 x + \alpha_1^2 - 3 = \left(x + \frac{\alpha_1}{2}\right)^2 + \frac{3\alpha_1^2 - 12}{4}.$$

Since α_2 and α_3 are roots of this equation, we conclude that $12 - 3\alpha_1^2$ is a square in F. Likewise we get that $12 - 3\alpha_2^2$ and $12 - 3\alpha_3^2$ are squares in F. Now observe

$$\prod_{k=1}^{3}(12 - 3\alpha_i^2) = -3^3 f(-2)f(2) = 36(3 - 3v^{-2})$$

$$\prod_{k=2}^{3}(12 - 3\alpha_i^2) = 3^2 g(-2)g(2) = 9(\alpha - 1)^2(\alpha + 1)^2$$

so $12 - 3\alpha_1^2$ is a square in F if and only if $3 - 3v^{-2}$ is a square in F. But $3 - 3v^{-2}$ is a rational number, and thus $3 - 3v^{-2}$ is a square in F iff it is a square in $\mathbb{Q}$ (otherwise its square root would generate a degree 2 subextension of F). So it suffices to determine the positive integers v for which $3v^2 - 3$ is a perfect square.

Setting $3v^2 - 3 = k^2$, we see that k must be a multiple of 3 so we can set $k = 3k'$ and get $v^2 - 3k'^2 = 1$. This is a Pell equation with solution (1,0) and smallest nontrivial solution $(v, k') = (2, 1)$, so all other solutions are given by the coefficients of 1 and $\sqrt{3}$ in $(2 + \sqrt{3})^m$ for $m \geq 1$. Computing recursively, we get the smallest four values of v are $1, 2, 7, 26$, for a final answer of $1 + 2 + 7 + 26 = \boxed{36}$.

Remark: This solution illustrates the general fact that an irreducible cubic has Galois group $\mathbb{Z}/3\mathbb{Z}$ if and only if its discriminant is a square in the base field.