Algebra & Number Theory Div. 1

1. How many multiples of 12 divide 12! and have exactly 12 divisors?

2. Suppose there are 160 pigeons and n holes. The 1st pigeon flies to the 1st hole, the 2nd pigeon flies to the 4th hole, and so on, such that the ith pigeon flies to the $(i^2 \bmod n)$th hole, where $k \bmod n$ is the remainder when k is divided by n. What is minimum n such that there is at most one pigeon per hole?

3. Let a and b be complex numbers such that $(a+1)(b+1) = 2$ and $(a^2+1)(b^2+1) = 32$. Compute the sum of all possible values of $(a^4+1)(b^4+1)$.

4. Let $f(x) = \frac{x^2}{8}$. Starting at the point $(7,3)$, what is the length of the shortest path that touches the graph of f, and then the x-axis?

5. Suppose f is a degree 42 polynomial such that for all integers $0 \le i \le 42$,

$$f(i) + f(43+i) + f(2 \cdot 43 + i) + \cdots + f(46 \cdot 43 + i) = (-2)^i$$

Find $f(2021) - f(0)$.

6. Find the remainder when

$$\left\lfloor \frac{149^{151} + 151^{149}}{22499} \right\rfloor$$

is divided by 10^4.

7. As a gift, Dilhan was given the number $n = 1^1 \cdot 2^2 \cdots 2021^{2021}$, and each day, he has been dividing n by 2021! exactly once. One day, when he did this, he discovered that, for the first time, n was no longer an integer, but instead a reduced fraction of the form $\frac{a}{b}$. What is the sum of all distinct prime factors of b?

8. There are integers v, w, x, y, z and real numbers $0 \le \theta < \theta' \le \pi$ such that

$$\cos 3\theta = \cos 3\theta' = v^{-1}, \qquad w + x\cos\theta + y\cos 2\theta = z\cos\theta'.$$

Given that $z \ne 0$ and v is positive, find the sum of the 4 smallest possible values of v.