Geometry Solutions

1. Let $PQRS$ be a square with side length 12. Point A lies on segment $\overline{QR}$ with $\angle QPA = 30^\circ$, and point B lies on segment $\overline{PQ}$ with $\angle SRB = 60^\circ$. What is AB?

   Proposed by Gunmay Handa

   Answer: $4\sqrt{6}$

   Note that since $PQ = 12$, $PA = 8\sqrt{3}$ so $QA = 4\sqrt{3}$. By similar reasoning and using that $\angle BRQ = 90-60 = 30^\circ$, $BQ = 4\sqrt{3}$. Then $AB = \sqrt{BQ^2 + QA^2} = \boxed{4\sqrt{6}}$.

2. Let ABC be a triangle. Points D and E are placed on $\overline{AC}$ in the order A, D, E, and C, and point F lies on $\overline{AB}$ with $EF \parallel BC$. Line segments $\overline{BD}$ and $\overline{EF}$ meet at X. If $AD = 1$, $DE = 3$, $EC = 5$, and $EF = 4$, compute FX.

   Proposed by Misha Ivkov and David Altizio

   Answer: $\frac{5}{8}$

   Solution. Since $\triangle AEF \sim \triangle ACB$, $BC = EF \cdot \frac{AC}{AE} = 4 \cdot \frac{9}{4} = 9$. Furthermore, since $\triangle DEX \sim \triangle DCB$, $XE = BC \cdot \frac{DE}{DC} = 9 \cdot \frac{3}{8} = \frac{27}{8}$. Thus $XF = EF - XE = \boxed{\frac{5}{8}}$.

3. Point A, B, C, and D form a rectangle in that order. Point X lies on CD, and segments $\overline{BX}$ and $\overline{AC}$ intersect at P. If the area of triangle BCP is 3 and the area of triangle PXC is 2, what is the area of the entire rectangle?

   Proposed by Josh Abrams

   Answer: 15

   Solution. Let Y be the intersection point of BX with AD. Compute $BP : PX = 3 : 2$, and furthermore

   $$\frac{3}{2} = \frac{BP}{PX} = \frac{BA}{CX} = \frac{CD}{CX}.$$

   So the area of $\triangle XDY$ is $\frac{5}{4}$, and hence the area of $\triangle ABY$ is $3 \cdot \frac{5}{4} = \frac{15}{4}$. Thus the area of $ABXD$ is 10, and we may conclude the answer of $\boxed{15}$.

4. Triangle ABC has a right angle at B. The perpendicular bisector of $\overline{AC}$ meets segment $\overline{BC}$ at D, while the perpendicular bisector of segment $\overline{AD}$ meets $\overline{AB}$ at E. Suppose CE bisects acute $\angle ACB$. What is the measure of angle ACB?

   Proposed by Daniel Li

   Answer: 36°

   Solution. The key observation is that quadrilateral $AEDC$ is cyclic. One can prove this by recalling that E is the midpoint of the arc AD of $\odot(ADC)$. (Alternatively, using the Angle Bisector Theorem yields

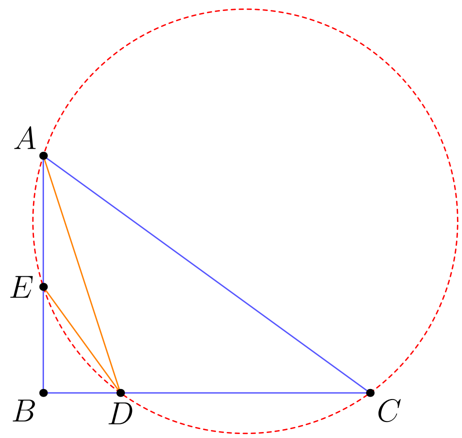
   $$\frac{BE}{ED} = \frac{BE}{EA} = \frac{BC}{CA},$$

   which gives $\triangle DEB \sim \triangle ACB$.)

   Now let $\angle ACE = \angle BCE = \alpha$. Then $\angle EAD = \angle ECD = \alpha$ while $\angle DAC = \angle ACD = 2\alpha$; in turn,

   $$90^\circ = \angle BAC + \angle ACB = 5\alpha.$$

   Solving yields $\alpha = 18^\circ$, so $\angle ACB = \boxed{36^\circ}$.



5. For every positive integer k, let $\mathbf{T}_k = (k(k+1), 0)$, and define $\mathcal{H}_k$ as the *homothety* centered at $\mathbf{T}_k$ with ratio $\frac{1}{2}$ if k is odd and $\frac{2}{3}$ is k is even. Suppose $P = (x, y)$ is a point such that

$$(\mathcal{H}_4 \circ \mathcal{H}_3 \circ \mathcal{H}_2 \circ \mathcal{H}_1)(P) = (20, 20).$$

What is $x + y$?

Proposed by Gunmay Handa

Answer: 256

Solution. The key claim is that the composition of two homotheties centered at A and B with ratios r and s is a homothety itself with ratio rs and center X lying on AB satisfying $\frac{XA}{XB} = \frac{(s-1)}{s(1-r)}$, where the ratio is directed.

Let P be a point in the plane, P' be the image of P under the homothety at A and P'' be the image of P' under the homothety at B; let $X = P'P'' \cap AB$. Then by Menelaus's theorem on $\triangle P'AB$ and transversal XP, we obtain

$$\frac{PP'}{PA} \cdot \frac{XA}{XB} \cdot \frac{BP''}{P''P} = -1 \implies (1-r) \cdot \frac{XA}{XB} \cdot \frac{s}{1-s} = -1 \implies \frac{XA}{XB} = \frac{(s-1)}{s(1-r)}$$

so the position of X does not depend on P. Moreover, we can apply Menelaus's theorem again on $\triangle PP'P''$ with transversal AB to conclude

$$\frac{XP''}{XP} \cdot \frac{AP}{AP'} \cdot \frac{BP'}{BP''} = 1 \implies \frac{XP''}{XP} = rs$$

so we can conclude the desired.

Using the claim, we conclude the composition of the four homotheties has ratio $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{9}$, and we can compute the center of $\mathcal{H}_2 \circ \mathcal{H}_1$ as $(4, 0)$, the center of $\mathcal{H}_4 \circ \mathcal{H}_3$ as $(16, 0)$, and the center of the composition of the two aforementioned composited homotheties as $(13, 0)$. Hence, x and y satisfy

$$\begin{aligned} x - 13 &= 9(20 - 13) \\ y - 0 &= 9(20 - 0) \end{aligned}$$

so $(x, y) = (76, 180)$ and the answer is $\boxed{256}$.

6. Two circles ω_A and ω_B have centers at points A and B respectively and intersect at points P and Q in such a way that A, B, P, and Q all lie on a common circle ω. The tangent to ω at P intersects ω_A and ω_B again at points X and Y respectively. Suppose $AB = 17$ and $XY = 20$. Compute the sum of the radii of ω_A and ω_B.

Proposed by David Altizio

Answer: $3\sqrt{51}$

Solution. First remark that since $PAQB$ is a kite, $\angle APB = \angle AQB$; combining this with $APBQ$ cyclic implies $\angle APB = \angle AQB = 90°$.

The key observation is that P is the midpoint of $\overline{XY}$. To prove this, let M and N be the feet of the perpendiculars from A and B respectively to XY. Then $\triangle MPA \sim \triangle PBA \sim \triangle NBP$, so

$$PM = PA \cdot \tfrac{PB}{AB} = PB \cdot \tfrac{PA}{AB} = PN.$$

Thus P is the midpoint of $\overline{MN}$, meaning it is also the midpoint of $\overline{XY}$.

To finish, let r_A and r_B be the radii of the circles ω_A and ω_B respectively. Then $r_A^2 + r_B^2 = 289$ by the Pythagorean Theorem. Furthermore, from the computation that established P was the midpoint of $\overline{MN}$ we have $r_A r_B = PM \cdot AB = 17 \cdot 5 = 85$. As a result,

$$r_A + r_B = \sqrt{r_A^2 + r_B^2 + 2r_A r_B} = \sqrt{17^2 + 2 \cdot 17 \cdot 5} = \boxed{3\sqrt{51}}.$$

7. In triangle ABC, points D, E, and F are on sides BC, CA, and AB respectively, such that $BF = BD = CD = CE = 5$ and $AE - AF = 3$. Let I be the incenter of ABC. The circumcircles of BFI and CEI intersect at $X \neq I$. Find the length of DX.

Proposed by Howard Halim

Answer: 3

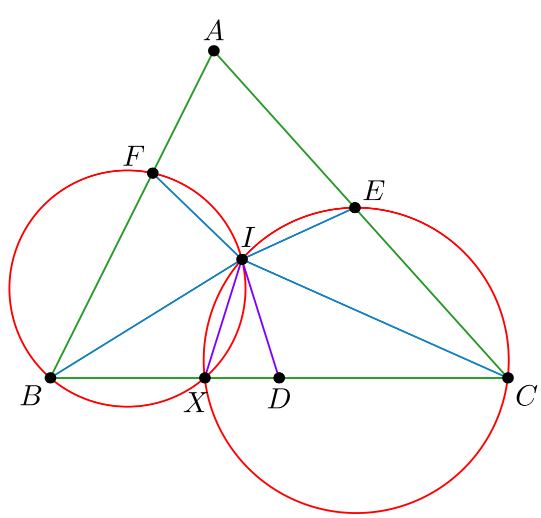
Solution. The key observation is that X lies on BC with $ID = IX$. To prove the first claim, observe via angle chasing that

$$\angle IXC = \angle BFI = \angle BDI,$$

where the last equality holds since BI is the perpendicular bisector of $\overline{DF}$. Similarly, $\angle IXB = \angle IDC$. Thus, $\angle BXI + \angle CXI = 180°$, whence $X \in BC$. The claim $IX = ID$ follows from the fact that I is the midpoint of, for example, arc $\widehat{EX}$. (The claim that X lies on BC also follows from Miquel's Theorem.)

Finally, remark that X and D are reflections across the tangency point T of the incircle with $\overline{BC}$, and so

$$\begin{aligned} DX &= TC - TB = (s-c) - (s-b) = b - c \\ &= (AE + EC) - (AF + FB) = AE - AF = \boxed{3}. \end{aligned}$$



8. Let $\mathcal{E}$ be an ellipse with foci F_1 and F_2. Parabola $\mathcal{P}$, having vertex F_1 and focus F_2, intersects $\mathcal{E}$ at two points X and Y. Suppose the tangents to $\mathcal{E}$ at X and Y intersect on the directrix of $\mathcal{P}$. Compute the eccentricity of $\mathcal{E}$.

   (A *parabola* $\mathcal{P}$ is the set of points which are equidistant from a point, called the *focus* of $\mathcal{P}$, and a line, called the *directrix* of $\mathcal{P}$. An *ellipse* $\mathcal{E}$ is the set of points P such that the sum $PF_1 + PF_2$ is some constant d, where F_1 and F_2 are the *foci* of $\mathcal{E}$. The *eccentricity* of $\mathcal{E}$ is defined to be the ratio F_1F_2/d.)

   Proposed by David Altizio

   Answer: $\frac{2+\sqrt{13}}{9}$

   Solution. Let T be the intersection point of the tangents to $\mathcal{E}$ at X and Y; note that $T \in F_1F_2$ by symmetry and that by the problem statement T lies on the directrix ℓ of $\mathcal{P}$. Recall that by the reflection property of ellipses, TX is the external angle bisector of $\angle F_1XF_2$. Thus the (Exterior) Angle Bisector Theorem implies $\frac{F_2X}{F_1X} = \frac{F_2T}{F_1T} = 2$.
   Denote by Q and R the projections of X onto ℓ and F_1F_2 respectively, and let $F_1X = x$ and $F_1F_2 = d$; then $TR = QX = XF_2 = 2x$. The Pythagorean Theorem applied to triangles F_1XR and F_2XR implies

$$(2x)^2 - x^2 = F_2X^2 - F_1X^2 = F_2R^2 - F_1R^2 = (2d - 2x)^2 - (2x - d)^2.$$

   Simplifying yields $3x^2 = 3d^2 - 4xd$. Thus the ratio $r := d/x$ satisfies the equality $3 = 3r^2 - 4r$, and so $r = \frac{2+\sqrt{13}}{3}$.

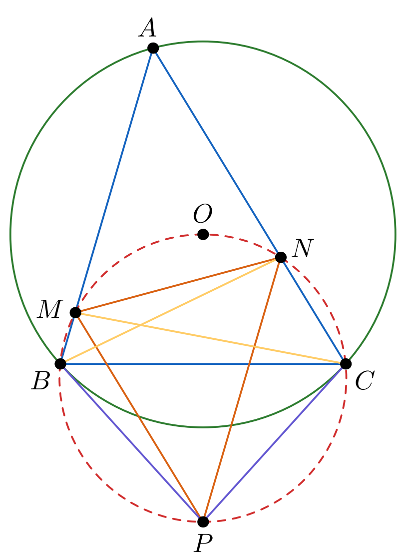
   Finally, the eccentricity of $\mathcal{E}$ is $\frac{d}{F_1X+F_2X} = \frac{d}{3x} = \boxed{\frac{2+\sqrt{13}}{9}}$.

9. In triangle ABC, points M and N are on segments AB and AC respectively such that $AM = MC$ and $AN = NB$. Let P be the point such that PB and PC are tangent to the circumcircle of ABC. Given that the perimeters of PMN and $BCNM$ are 21 and 29 respectively, and that $PB = 5$, compute the length of BC.

   Proposed by Howard Halim

   Answer: $\frac{200}{21}$

   Solution. Observe that $\angle BAC = \angle PBC$, so triangles $\triangle ANB$ and $\triangle CPB$ are similar to each other. It follows by spiral similarity that $\triangle BNP \sim \triangle BAC$. In turn, $\angle BPN = \angle BCN$, whence $BNCP$ is cyclic. Analogous reasoning implies M lies on this circle as well.



   Furthermore, triangles BNP and PCM are actually congruent, since they are both similar to $\triangle BAC$ and have common side $BP = CP$. This means $BN = MP$ and $CM = NP$, so quadrilaterals $BMNP$ and $CNMP$ are both isosceles trapezoids. In turn, $MN = BP = CP = 5$ and quadrilateral $AMPN$ is a parallelogram.

Since the perimeter of $\triangle AMN$ is 21, the perimeter of $\triangle ABC$ is $21 + 29 - 2 \cdot 5 = 40$. Hence triangles ANM and ABC are similar with a scale factor of $\frac{21}{40}$, implying

$$BC = MN \cdot \frac{40}{21} = 5 \cdot \frac{40}{21} = \boxed{\frac{200}{21}}.$$

10. Four copies of an acute scalene triangle $\mathcal{T}$, one of whose sides has length 3, are joined to form a tetrahedron with volume 4 and surface area 24. Compute the largest possible value for the circumradius of $\mathcal{T}$.

Proposed by Misha Ivkov and David Altizio

Answer: $\sqrt{4+\sqrt{3}}$

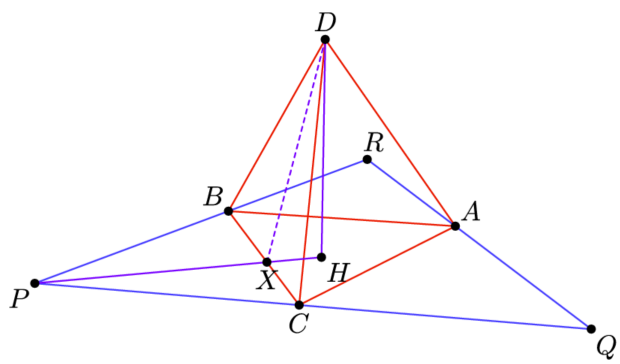
Solution. Denote the tetrahedron $\mathcal{T}$ by $ABCD$, where we set $BC = 3$ without loss. The key idea is to *unfold* the tetrahedron into a net, transforming it into triangle PQR as shown below. This leads into a key lemma.

Lemma 1 (1999 AIME #15, etc.)**.** *The foot H of the perpendicular from D to plane ABC is the orthocenter of $\triangle PQR$.*

Proof. Note that

$$BH^2 - CH^2 = BD^2 - DC^2 = BP^2 - PC^2;$$

the first equality follows from the Pythagorean Theorem while the second comes from the equalities $DB = BP$ and $DC = CP$. Thus the Perpendicularity Lemma tells us $PH \perp BC$, so H lies on the altitude from P. Applying this reasoning cyclically yields the desired. $\square$



Since the surface area of $\mathcal{T}$ is 24, the area of $\triangle ABC$ is 6, and so via the formula $V = \frac{1}{3}bh$ we may deduce $DH = 2$. Let X denote the intersection of PH with BC, i.e. the foot of the altitude from P to BC. By a similar area argument as above, we deduce $PX = 4$, and via the definition of reflection, $XD = 4$ as well. So $XH = \sqrt{4^2 - 2^2} = 2\sqrt{3}$, implying $PH = 4 + 2\sqrt{3}$.

Finally, let R denote the circumradius of $\triangle ABC$. Then the circumradius of $\triangle PQR$ is $2R$. Thus

$$2(2R)^2 = PH^2 + RQ^2 = (4 + 2\sqrt{3})^2 + 6^2 = 8(4 + \sqrt{3}),$$

so $R = \boxed{\sqrt{4+\sqrt{3}}}$.

Note that this tetrahedron actually exists when $\mathcal{T}$ has side lengths approximately 3, 4.021, and 4.763.

11. (Estimation) Gunmay picks 6 points uniformly at random in the unit square. If p is the probability that their convex hull is a hexagon, estimate p in the form $0.abcdef$ where a, b, c, d, e, f are decimal digits. (A *convex combination* of points $x_1, x_2, \ldots, x_n$ is a point of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ with $0 \le \alpha_i \le 1$ for all i and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. The *convex hull* of a set of points X is the set of all possible convex combinations of all subsets of X.)

Proposed by Max Aires

Answer: 0.122327