CMIMC 2020 Geometry Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Geometry

1. Let $PQRS$ be a square with side length 12. Point A lies on segment $\overline{QR}$ with $\angle QPA = 30^\circ$, and point B lies on segment $\overline{PQ}$ with $\angle SRB = 60^\circ$. What is AB?

2. Let ABC be a triangle. Points D and E are placed on $\overline{AC}$ in the order A, D, E, and C, and point F lies on $\overline{AB}$ with $EF \parallel BC$. Line segments $\overline{BD}$ and $\overline{EF}$ meet at X. If $AD = 1$, $DE = 3$, $EC = 5$, and $EF = 4$, compute FX.

3. Point A, B, C, and D form a rectangle in that order. Point X lies on CD, and segments $\overline{BX}$ and $\overline{AC}$ intersect at P. If the area of triangle BCP is 3 and the area of triangle PXC is 2, what is the area of the entire rectangle?

4. Triangle ABC has a right angle at B. The perpendicular bisector of $\overline{AC}$ meets segment $\overline{BC}$ at D, while the perpendicular bisector of segment $\overline{AD}$ meets $\overline{AB}$ at E. Suppose CE bisects acute $\angle ACB$. What is the measure of angle ACB?

5. For every positive integer k, let $\mathbf{T}_k = (k(k+1), 0)$, and define $\mathcal{H}_k$ as the homothety centered at $\mathbf{T}_k$ with ratio $\frac{1}{2}$ if k is odd and $\frac{2}{3}$ is k is even. Suppose $P = (x, y)$ is a point such that
$$(\mathcal{H}_4 \circ \mathcal{H}_3 \circ \mathcal{H}_2 \circ \mathcal{H}_1)(P) = (20, 20).$$
What is $x + y$?

   (A *homothety* $\mathcal{H}$ with nonzero ratio r centered at a point P maps each point X to the point Y on ray $\overrightarrow{PX}$ such that $PY = rPX$.)

6. Two circles ω_A and ω_B have centers at points A and B respectively and intersect at points P and Q in such a way that A, B, P, and Q all lie on a common circle ω. The tangent to ω at P intersects ω_A and ω_B again at points X and Y respectively. Suppose $AB = 17$ and $XY = 20$. Compute the sum of the radii of ω_A and ω_B.

7. In triangle ABC, points D, E, and F are on sides BC, CA, and AB respectively, such that $BF = BD = CD = CE = 5$ and $AE - AF = 3$. Let I be the incenter of ABC. The circumcircles of BFI and CEI intersect at $X \neq I$. Find the length of DX.

8. Let $\mathcal{E}$ be an ellipse with foci F_1 and F_2. Parabola $\mathcal{P}$, having vertex F_1 and focus F_2, intersects $\mathcal{E}$ at two points X and Y. Suppose the tangents to $\mathcal{E}$ at X and Y intersect on the directrix of $\mathcal{P}$. Compute the eccentricity of $\mathcal{E}$.

   (A *parabola* $\mathcal{P}$ is the set of points which are equidistant from a point, called the *focus* of $\mathcal{P}$, and a line, called the *directrix* of $\mathcal{P}$. An *ellipse* $\mathcal{E}$ is the set of points P such that the sum $PF_1 + PF_2$ is some constant d, where F_1 and F_2 are the *foci* of $\mathcal{E}$. The *eccentricity* of $\mathcal{E}$ is defined to be the ratio F_1F_2/d.)

9. In triangle ABC, points M and N are on segments AB and AC respectively such that $AM = MC$ and $AN = NB$. Let P be the point such that PB and PC are tangent to the circumcircle of ABC. Given that the perimeters of PMN and $BCNM$ are 21 and 29 respectively, and that $PB = 5$, compute the length of BC.

10. Four copies of an acute scalene triangle $\mathcal{T}$, one of whose sides has length 3, are joined to form a tetrahedron with volume 4 and surface area 24. Compute the largest possible value for the circumradius of $\mathcal{T}$.

11. (Estimation) Gunmay picks 6 points uniformly at random in the unit square. If p is the probability that their convex hull is a hexagon, estimate p in the form $0.abcdef$ where a, b, c, d, e, f are decimal digits. (A *convex combination* of points $x_1, x_2, \ldots, x_n$ is a point of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ with $0 \leq \alpha_i \leq 1$ for all i and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. The *convex hull* of a set of points X is the set of all possible convex combinations of all subsets of X.)