CMIMC 2020

Combinatorics and Computer Science Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the middle of events (5:15 PM).

Combinatorics and Computer Science

1. The intramural squash league has 5 players, namely Albert, Bassim, Clara, Daniel, and Eugene. Albert has played one game, Bassim has played two games, Clara has played 3 games, and Daniel has played 4 games. Assuming no two players in the league play each other more than one time, how many games has Eugene played?

2. David is taking a true/false exam with 9 questions. Unfortunately, he doesn't know the answer to any of the questions, but he does know that exactly 5 of the answers are True. In accordance with this, David guesses the answers to all 9 questions, making sure that exactly 5 of his answers are True. What is the probability he answers at least 5 questions correctly?

3. Consider a 1-indexed array that initially contains the integers 1 to 10 in increasing order.

   The following action is performed repeatedly (any number of times):

```
def action():
    Choose an integer n between 1 and 10 inclusive
    Reverse the array between indices 1 and n inclusive
    Reverse the array between indices n+1 and 10 inclusive (If n = 10, we do nothing)
```

   How many possible orders can the array have after we are done with this process?

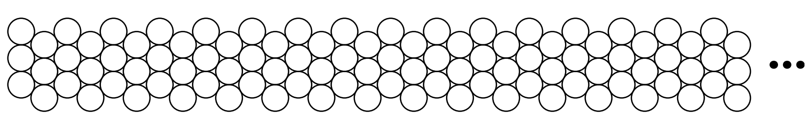
4. The continent of Trianglandia is an equilateral triangle of side length 9, divided into 81 triangular countries of side length 1. Each country has the resources to choose at most 1 of its 3 sides and build a "wall" covering that entire side. However, since all the countries are at war, no two countries are willing to have their walls touch, even at a corner. What is the maximum number of walls that can be built in Trianglandia?

5. Seven cards numbered 1 through 7 lay stacked in a pile in ascending order from top to bottom (1 on top, 7 on bottom). A shuffle involves picking a random card *of the six not currently on top* and putting it on top. The relative order of all the other cards remains unchanged. Find the probability that, after 10 shuffles, 6 is higher in the pile than 3.

6. The nation of CMIMCland consists of 8 islands, none of which are connected. Each citizen wants to visit the other islands, so the government will build bridges between the islands. However, each island has a volcano that could erupt at any time, destroying that island and any bridges connected to it. The government wants to guarantee that after any eruption, a citizen from any of the remaining 7 islands can go on a tour, visiting each of the remaining islands exactly once and returning to their home island (only at the end of the tour). What is the minimum number of bridges needed?

7. Consider a complete graph of 2020 vertices. What is the least number of edges that need to be marked such that each triangle (3-vertex subgraph) has an odd number of marked edges?

8. Catherine has a plate containing 300 circular crumbling mooncakes, arranged as follows:



   (This continues for 100 total columns). She wants to pick some of the mooncakes to eat, however whenever she takes a mooncake all adjacent mooncakes will be destroyed and cannot be eaten. Let M be the maximal number of mooncakes she can eat, and let n be the number of ways she can pick M mooncakes to eat (Note: the order in which she picks mooncakes does not matter). Compute the ordered pair (M, n).

9. Let $\Gamma = \{\varepsilon, 0, 00, \ldots\}$ be the set of all finite strings consisting of only zeroes. We consider *six-state unary DFAs* $D = (F, q_0, \delta)$ where F is a subset of $Q = \{1, 2, 3, 4, 5, 6\}$, not necessarily strict and possibly empty; $q_0 \in Q$ is some *start state*; and $\delta : Q \to Q$ is the *transition function*. For each such DFA D, we associate a set $F_D \subseteq \Gamma$ as the set of all strings $w \in \Gamma$ such that

$$\underbrace{\delta(\cdots(\delta(q_0))\cdots)}_{|w| \text{ applications}} \in F,$$

We say a set $\mathcal{D}$ of DFAs is *diverse* if for all $D_1, D_2 \in \mathcal{D}$ we have $F_{D_1} \neq F_{D_2}$. What is the maximum size of a diverse set?

10. Define a string to be doubly palindromic if it can be split into two (non-empty) parts that are read the same both backwards and forwards. For example hannahhuh is doubly palindromic as it can be split into hannah and huh. How many doubly palindromic strings of length 9 using only the letters $\{a, b, c, d\}$ are there?

11. (Estimation) Max flips 2020 fair coins. Let the probability that there are at most 505 heads be p. Estimate $-\log_2(p)$ to 5 decimal places, in the form $x.abcde$ where x is a positive integer and a, b, c, d, e are decimal digits.