CMIMC 2020

Algebra and Number Theory Solutions

1. Suppose x is a real number such that $x^2 = 10x + 7$. Find the unique ordered pair of integers (m, n) such that $x^3 = mx + n$.

   Proposed by Vijay Srinivasan

   Answer: $(107, 70)$

   Solution. We have

   $$x^3 = x \cdot x^2 = x(10x + 7) = 10x^2 + 7x = 10(10x + 7) + 7x = 107x + 70$$

   giving the answer $\boxed{(107, 70)}$. If there were another possible ordered pair, this would imply x is rational, but since $(x - 5)^2 = 32$, this is clearly not the case.

2. Find the unique real number c such that the polynomial $x^3 + cx + c$ has exactly two real roots.

   Proposed by Vijay Srinivasan

   Answer: $-\frac{27}{4}$

   Solution. If there are two real roots, then the third root must also be real, and one of the roots is a double root. Thus we have

   $$x^3 + cx + c = (x - a)^2(x - b) = x^3 - (2a + b)x + (a^2 + 2ab)x - a^2b$$

   This gives $b = -2a$, equating the other coefficients gives $c = -3a^2$ and $c = 2a^3$. So $-3a^2 = 2a^3$, giving $a = 0$ or $a = -3/2$. If $a = 0$, then $b = 0$, giving only one real root, so this can't occur. So $a = -3/2$ and $\boxed{c = -27/4}$.

3. Call a number "Sam-azing" if it is equal to the sum of its digits times the product of its digits. The only two three-digit Sam-azing numbers are n and $n + 9$. Find n.

   Proposed by Sam Delatore

   Answer: 135

   Solution. We present two solutions.

   First solution, by Partial Guesswork Observe that one of the two Sam-azing numbers, either n or $n + 9$, is odd. Since each of the digits of this number divides it, it follows that each of the digits is odd. Testing a few small cases yields that 135 is Sam-azing.

   Thus, the other Sam-azing number is either 126 or 144; of these, only 144 works, so $n = \boxed{135}$.

   Solution 2, by Direct Reasoning First observe that no positive Sam-azing integer can have a zero digit; call this observation (†). With this in mind, we make two crucial observations:

   - Notice that n cannot have a units digit of either zero or one (the first one contradicts (†) for n, the second contradicts (†) for $n + 9$). Thus n has a units digit of at least two. As a consequence, the units and tens digits of n and $n + 9$ are both different.
   - Let a be the hundreds digit of n. Then

     $$n + 9 \notin [100(a + 1), 100(a + 1) + 9],$$

     since otherwise $n + 9$ has a zero digit, contradicting (†). It follows that $n + 9 \leq 100a + 90$. As a consequence, the hundreds digits of n and $n + 9$ are identical.

Combining both observations yields that

$$n = \overline{abc} \quad \text{and} \quad n + 9 = \overline{a(b+1)(c-1)}$$

for some nonzero digits a, b, and c.

So

$$9 = a(b+1)(c-1)(a+b+c) - abc(a+b+c) = a(c-b-1)(a+b+c).$$

Since a, b, and c are positive, and furthermore $c > 1$, we must have $a+b+c \geq 4$, so $a+b+c = 9$. It follows that $a = c-b-1 = 1$. Solving this system of equations yields $b = 3$ and $c = 5$, so $n = \boxed{135}$.

4. For all real numbers x, let $P(x) = 16x^3 - 21x$. What is the sum of all possible values of $\tan^2\theta$, given that θ is an angle satisfying

$$P(\sin\theta) = P(\cos\theta)?$$

Proposed by David Altizio

Answer: $\frac{231}{25}$

Solution. Rewrite the given equality as

$$16\sin^3\theta - 21\sin\theta = 16\cos^3\theta - 21\cos\theta, \quad \text{or} \quad 16(\sin^3\theta - \cos^3\theta) = 21(\sin\theta - \cos\theta).$$

If $\sin\theta = \cos\theta$, equality is trivially satisfied, and hence $\tan\theta = 1$. Otherwise, we may divide both sides by $\sin\theta - \cos\theta$ to get

$$21 = 16(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta) = 16(1 + \sin\theta\cos\theta), \quad \text{or} \quad \sin\theta\cos\theta = \tfrac{5}{16}.$$

Now remark that there is one positive solution $(x, y) = (\sin\theta, \cos\theta)$ to the system of equations $x^2 + y^2 = 1$ and $xy = \frac{5}{16}$ up to permutation. Thus the two values of $\tan\theta$ are $\frac{x}{y}$ and $\frac{y}{x}$, and so the sum of the values of $\tan^2\theta$ is

$$\frac{x^2}{y^2} + \frac{y^2}{x^2} = \frac{(x^2+y^2)^2}{(xy)^2} - 2 = \frac{256}{25} - 2 = \frac{206}{25}.$$

Pulling the two cases together gives the requested answer of $\boxed{\frac{231}{25}}$.

5. Let $f(x) = 2^x + 3^x$. For how many integers $1 \leq n \leq 2020$ is $f(n)$ relatively prime to all of $f(0), f(1), \ldots, f(n-1)$?

Proposed by Adam Bertelli

Answer: 11

Solution. We claim that $f(n)$ is relatively prime to all of $f(0), \ldots, f(n-1)$ if and only if n is a power of 2 (Note that $2^0 = 1$ is a power of two). Since $2^{10} = 1024$ while $2^{11} = 2048$, we get a total of $\boxed{11}$ values of n.

We first prove the "only if" direction. Suppose n is *not* a power of 2, so that $n = 2^k \cdot m$ for some odd integer m. Observe that

$$f(n) = 2^{2^k\cdot m} + 3^{2^k\cdot m} = (2^{2^k})^m + (3^{2^k})^m = (2^{2^k} + 3^{2^k}) \cdot M = f(2^k) \cdot M,$$

where M is some positive integer depending on k and m. Thus $f(2^k)$ is not relatively prime to $f(n)$, implying n does not satisfy the condition.

The "if" condition is somewhat more difficult. The solution proceeds in two lemmas.

Lemma 1. *For any positive integers m and n with $m < n$,*

$$\gcd(2^{2^m} + 3^{2^m}, 2^{2^n} + 3^{2^n}) = 1.$$

Proof. Suppose for contradiction that the greatest common divisor d is not 1, and let p be a prime dividing d. Observe that $2^{2^m} \equiv -3^{2^m} \pmod{p}$, and squaring both sides repeatedly eventually yields $2^{2^n} \equiv 3^{2^n} \pmod{p}$. (In particular, the negative sign goes away since we need to square at least once.) Thus, p divides both $3^{2^n} + 2^{2^n}$ and $3^{2^n} - 2^{2^n}$, implying that p divides both 3^{2^n} and 2^{2^n}. This is impossible. □

Lemma 2. *Suppose a and b are nonnegative integers such that there exists a prime p dividing both $f(a)$ and $f(b)$. Then p divides either $3^d + 2^d$ or $3^d - 2^d$, where $d = \gcd(a, b)$.*

Proof. Without loss of generality let $a > b$, so that $a = bq + r$ by the Division Algorithm. Then, since p divides both $3^a + 2^a$ and $3^b + 2^b$, it must also divide

$$3^a + 2^a - 2^r(3^{bq} + 2^{bq}) = 3^{bq}(3^r - 2^r).$$

But one can check $p \neq 3$, so p divides $3^r - 2^r$. Continuing inductively (with appropriate sign changes whenever necessary) proves the result. □

Now suppose $n = 2^r$ for some nonnegative integer k, and let $m < n$. let p be a prime dividing both $f(n)$ and $f(m)$. Observe that $\gcd(n, m) = 2^s$ for some nonnegative integer s, so by Lemma 2, p divides either $3^{2^s} + 2^{2^s}$ or $3^{2^s} - 2^{2^s}$. The former case is immediately ruled out by Lemma 1. For the latter case, we instead note the factorization

$$3^{2^s} - 2^{2^s} = (3+2)(3^2+2^2)\cdots(3^{2^{s-1}} + 2^{2^{s-1}}).$$

Since p divides this product, p must divide $3^{2^t} + 2^{2^t}$ for some $t \leq s - 1$. Again, this is a contradiction by Lemma 1. Hence such a prime does not exist and $\gcd(f(n), f(m)) = 1$. Since m was arbitrary, we deduce that n works.

6. Find all pairs of integers (x, y) such that $x \geq 0$ and

$$(6^x - y)^2 = 6^{x+1} - y.$$

Proposed by Dilhan Salgado

Answer: $(1, 0), (1, 11), (4, 1215), (4, 1376)$

Solution. Let $a = 6^x - y$. Then

$$a(a-1) = a^2 - a = 6^{x+1} - y - 6^x + y = 5 \cdot 6^x = 2^x \cdot 3^x \cdot 5.$$

Notice that 2^x divides $a(a-1)$. However, a and $a-1$ are relatively prime, so by Euclid's Lemma, 2^x must divide either a or $a-1$. Similarly, 3^x must also divide either a or $a-1$. This yields two cases.

Case 1: 2^x and 3^x both divide the same factor. Then $\{a, a-1\} = \{6^x, 5\}$, and so 6^x and 5 must differ by 1. The only way this can happen is if $x = 1$; in this case, a is either -5 or 6.

Case 2: 2^x and 3^x divide different factors. In this case, we need $5 \cdot 2^x$ and 3^x to differ by 1. (The other case is clearly not possible, since 2^x and $3^x \cdot 5$ always differ by greater than 1.) The only way this can happen is if $x = 4$; in this case, $a = 81$ or $a = -80$.

Solving both cases we get the 4 solutions $\boxed{(1, 0), (1, 11), (4, 1215), \text{ and } (4, 1376)}$.

7. Compute the positive difference between the two real solutions to the equation

$$(x-1)(x-4)(x-2)(x-8)(x-5)(x-7) + 48\sqrt{3} = 0.$$

Proposed by Misha Ivkov

Answer: $\sqrt{25+8\sqrt{3}}$

Solution. We proceed by making a series of substitutions. It is possible to solve this problem with fewer substitutions, but we choose to present the solution below for motivational purposes.

First remark that, by grouping factors together, the equation rewrites as

$$(x^2-9x+8)(x^2-9x+14)(x^2-9x+20)+48\sqrt{3}=0.$$

Now, to exploit symmetry, set $z=x^2-9x+14$; then the equation becomes

$$z(z-6)(z+6)+48\sqrt{3}=0.$$

Now let $w=\frac{z}{6}$ (to divide out the common factors of 6) to get

$$w^3-w+\frac{2}{3\sqrt{3}}=0$$

Finally, let $v=w\sqrt{3}$, so that then the equation becomes

$$v^3-3v+2=0.$$

By Rational Root Theorem, the solutions are $v=-2$ and $v=1$. Propagating this information up gives the desired solution.

8. Let $f:\mathbb{N}\to(0,\infty)$ satisfy $\prod_{d|n} f(d)=1$ for every n which is not prime. Determine the maximum possible number of n with $1\le n\le 100$ and $f(n)\neq 1$.

Proposed by Vijay Srinivasan

Answer: 82

Solution. Let $g(n)=\log f(n)$ and $h(n)=\sum_{d|n} g(d)$; the condition is equivalent to $h(n)=0$ for every n which is not prime. By Möbius inversion,

$$g(n)=\sum_{d|n}\mu(d)h\left(\frac{n}{d}\right).$$

We can make the above term nonzero as long as there is some d for which d is squarefree (so that $\mu(d)$ is nonzero) and $\frac{n}{d}$ is prime. To guarantee that $g(n)$ is nonzero whenever possible, make the values of h at primes $\mathbb{Q}$-linearly independent (e.g. $h(p)=\sqrt{p}$ for primes works). Therefore we can have $f(n)\neq 1$ if and only if n is the product of a prime and a squarefree number.

To count the number of n for which $f(n)\neq 1$, we will instead count the number of n for which $f(n)=1$. Note that $f(1)=1$. Next, if n is divisible by the cube of a prime, then $f(n)=1$. This includes multiples of 8 and 27 of which there are $12+3=15$ (with no overlap). The remaining cases are numbers which are divisible by the squares of two distinct primes but not divisible by the cube of a prime. For $n\le 100$, this only includes $2^2 3^2=36$ and $2^2 5^2=100$. The total number of n for which $f(n)=1$ is $1+15+2=18$, so there are $\boxed{82}$ values of n for which $f(n)\neq 1$.

9. Let $p=10009$ be a prime number. Determine the number of ordered pairs of integers (x,y) such that $1\le x,y\le p$ and $x^3-3xy+y^3+1$ is divisible by p.

Proposed by Vijay Srinivasan

Answer: 30024

Solution. Note that if ω is a primitive cube root of unity, then over reals we have the factorization

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z).$$

Since $10009 \equiv 1 \pmod 3$, there exists a primitive cube root of unity modulo 10009 (which we will also denote ω), and the same factorization applies. Specializing to $z = 1$ yields

$$x^3 + y^3 + 1 - 3xy = (x + y + 1)(x + \omega y + \omega^2)(x + \omega^2 y + \omega).$$

These are three linear expressions, and each pair of an equation and a value of y results in a unique value of x to make the expression 0. For each pair of linear expressions, there is a unique value of y for which the solution for x will coincide. We may check that these overlaps occur at $y = 1, \omega, \omega^2$, so the number of pairs (x, y) is $3p - 3 = \boxed{30024}$.

10. We call a polynomial P *square-friendly* if it is monic, has integer coefficients, and there is a polynomial Q for which $P(n^2) = P(n)Q(n)$ for all integers n. We say P is *minimally square-friendly* if it is square-friendly and cannot be written as the product of nonconstant, square-friendly polynomials. Determine the number of nonconstant, minimally square-friendly polynomials of degree at most 12.

Proposed by Vijay Srinivasan

Answer: 18

Solution. Say P is square-friendly. We claim that each irreducible factor of P must be either x or a cyclotomic polynomial Φ_n for some n. Let α be a nonzero root of P. Note that the condition on P means that α^{2^n} is a root for each positive integer n. If all of these are unique, then P has infinitely many roots, which is impossible, so $\alpha^{2^m} = \alpha^{2^n}$ for some m, n; in particular α is a root of unity. It follows that P is uniquely expressible as a product of cyclotomic polynomials and a power of x. Further, it is clear that if P is minimally square-friendly, then there are no powers of x unless $P(x) = x$.

We now claim that if P is square-friendly and Φ_{2k} divides P, then so does Φ_k. Indeed, the roots of Φ_{2k} are the primitive $2k$-th roots of unity; their squares are then the primitive k-th roots of unity and so necessarily Φ_k divides P.

This shows that if $n = 2^k m$ where m is odd, and Φ_n divides P, so does the product

$$\Psi_n := \Phi_m \Phi_{2m} \Phi_{4m} \cdots \Phi_{2^{k-1}m} \Phi_{2^k m = n}.$$

We claim that Ψ_n is square-friendly. For this, we will need two facts, collected in the following lemma.

Lemma: If b is odd, then $\Phi_b(x^2) = \Phi_b(x)\Phi_{2b}(x)$ and if b is even, then $\Phi_b(x^2) = \Phi_{2b}(x)$.

Proof: When b is even, the map $\alpha \mapsto \alpha^2$ is easily seen to be 2-to-1 from the roots of Φ_{2b} to the roots of Φ_b, so this finishes the even case. When b is odd, this map is instead 1-to-1, so that $\Phi_{2b}(x) \mid \Phi_b(x^2)$. We also have that the map $\alpha \mapsto \alpha^2$ is a bijection on the roots of Φ_b since b is odd, so $\Phi_b \mid \Phi_b(x^2)$. Since Φ_b and Φ_{2b} are coprime, we see by comparing degrees that $\Phi_b(x)\Phi_{2b}(x) = \Phi_b(x^2)$. Q.E.D.

Given the lemma, we see that

$$\Psi_n(x) = \Phi_m(x^2)\Phi_{2m}(x^2) \cdots \Phi_{2^{k-1}m}(x^2)$$

and so we certainly have $\Psi_n(x) \mid \Psi_n(x^2)$. In fact, we see that $\Psi_n(x^2) = \Psi_n(x)\Phi_{2n}(x)$.

We now show that any minimally square-friendly P (besides $P(x) = x$) is equal to some Ψ_n. Let n be the largest integer for which $\Phi_n \mid P$. Then our previous results show that $\Psi_n \mid P$ and that $\Psi_n(x^2) = \Psi_n(x)\Phi_{2n}(x)$. So if $P = \Psi_n Q$ then we have that $P(x^2) = \Psi_n(x)\Phi_{2n}(x)Q(x^2)$. It follows that

$$\Psi_n(x)Q(x) \mid \Psi_n(x)\Phi_{2n}(x)Q(x^2)$$

which implies that $Q(x) \mid \Phi_{2n}(x)Q(x^2)$. But since n was chosen to be maximal such that Φ_n divides P, it follows that Q and Φ_{2n} are coprime and thus $Q(x) \mid Q(x^2)$. Thus Q is square-friendly. It follows by P being minimally square-friendly that $Q = 1$, so $P = \Psi_n$.

Now we can easily compute

$$\deg \Psi_n = \phi(m)(1 + 1 + 2 + \cdots + 2^{k-1}) = 2^k \phi(m)$$

which otherwise stated says that $\deg \Psi_n = \phi(n)$ if n is odd and $2\phi(n)$ if n is even.

So we want to find the number of odd n with $\phi(n) \le 12$ and the number of even n with $\phi(n) \le 6$. This is equivalent to finding the number of n with $\phi(n) \le 6$ and the number of odd n with $7 \le \phi(n) \le 12$.

It is not hard to compute that

$$\phi^{-1}(\{1,2,4,6\}) = \{1,2,3,4,5,6,7,8,9,10,12,14,18\}.$$

Now if n is odd and $\phi(n) = 8$, it is easy to see that $n = 15$. If n is odd and $\phi(n) = 10$ then we need $n = 11$. If n is odd and $\phi(n) = 12$ then we can compute $n = 13$ or $n = 21$. So the set of admissible n is given by $1 \le n \le 15$, $n = 18$, and $n = 21$, for a total of 17 distinct Ψ_n. Remembering that x is also minimally square-free, there are a total of $\boxed{18}$ such polynomials.

11. (Estimation) Vijay picks two random distinct primes $1 \le p, q \le 10^4$. Let r be the probability that $3^{2205403200} \equiv 1 \bmod pq$. Estimate r in the form $0.abcdef$, where a, b, c, d, e, f are decimal digits.

Proposed by Misha Ivkov

Answer: $\frac{8789}{377303} \approx 0.0232942754$