CMIMC 2020

Algebra and Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes and one estimation question. Each of the short-answer questions is worth points depending on its difficulty, and the estimation question will be used to break ties. **If you do not write an estimate for estimation, you will be placed last in tiebreaking.**
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Algebra and Number Theory

1. Suppose x is a real number such that $x^2 = 10x + 7$. Find the unique ordered pair of integers (m, n) such that $x^3 = mx + n$.

2. Find the unique real number c such that the polynomial $x^3 + cx + c$ has exactly two real roots.

3. Call a number "Sam-azing" if it is equal to the sum of its digits times the product of its digits. The only two three-digit Sam-azing numbers are n and $n + 9$. Find n.

4. For all real numbers x, let $P(x) = 16x^3 - 21x$. What is the sum of all possible values of $\tan^2\theta$, given that θ is an angle satisfying
$$P(\sin\theta) = P(\cos\theta)?$$

5. Let $f(x) = 2^x + 3^x$. For how many integers $1 \le n \le 2020$ is $f(n)$ relatively prime to all of $f(0), f(1), \dots, f(n-1)$?

6. Find all pairs of integers (x, y) such that $x \ge 0$ and
$$(6^x - y)^2 = 6^{x+1} - y.$$

7. Compute the positive difference between the two real solutions to the equation
$$(x-1)(x-4)(x-2)(x-8)(x-5)(x-7) + 48\sqrt{3} = 0.$$

8. Let $f : \mathbb{N} \to (0, \infty)$ satisfy $\prod_{d|n} f(d) = 1$ for every n which is not prime. Determine the maximum possible number of n with $1 \le n \le 100$ and $f(n) \ne 1$.

9. Let $p = 10009$ be a prime number. Determine the number of ordered pairs of integers (x, y) such that $1 \le x, y \le p$ and $x^3 - 3xy + y^3 + 1$ is divisible by p.

10. We call a polynomial P *square-friendly* if it is monic, has integer coefficients, and there is a polynomial Q for which $P(n^2) = P(n)Q(n)$ for all integers n. We say P is *minimally square-friendly* if it is square-friendly and cannot be written as the product of nonconstant, square-friendly polynomials. Determine the number of nonconstant, minimally square-friendly polynomials of degree at most 12.

11. (Estimation) Vijay picks two random distinct primes $1 \le p, q \le 10^4$. Let r be the probability that $3^{2205403200} \equiv 1 \bmod pq$. Estimate r in the form $0.abcdef$, where a, b, c, d, e, f are decimal digits.