Team Solutions

1. David recently bought a large supply of letter tiles. One day he arrives back to his dorm to find that some of the tiles have been arranged to read Central Michigan University. What is the smallest number of tiles David must remove and/or replace so that he can rearrange them to read Carnegie Mellon University?

   Proposed by David Altizio

   Answer: 5

   Solution. Notice that David need not adjust any letters from the word University, since these also appear in the desired phrase. In general, it suffices to determine which letters are common to both phrases; the unmatched letters from Central Michigan University must then be removed or replaced with letters from Carnegie Mellon University. Doing this results in the following figure, where the top row represents letters from the former expression and the bottom row represents letters from the latter expression.

   CenralMign tchia
   CenralMign eelo

   It follows that David must replace $\boxed{5}$ letters to reach his goal.

2. Determine the number of ordered pairs of positive integers (m, n) with $1 \leq m \leq 100$ and $1 \leq n \leq 100$ such that
$$\gcd(m+1, n+1) = 10\gcd(m, n).$$

   Proposed by David Altizio

   Answer: 52

   Solution. The crucial claim is that $\gcd(m, n) = 1$. Indeed, suppose not, and let $\gcd(m, n) = k > 1$. Then $k \mid \gcd(m+1, n+1)$ as well, so $k \mid m$ and $k \mid m+1$ simultaneously. This contradicts $k > 1$. Hence $\gcd(m, n) = 1$, which implies $\gcd(m+1, n+1) = 10$.

   Now as $m+1$ and $n+1$ are both multiples of 10, they must come from the set
$$\{9, 19, 29, 39, 49, 59, 69, 79, 89, 99\}.$$

   A quick inspection reveals that 9, 39, 69, and 99 are multiples of 3; 49 is a power of 7; and the remaining five integers are prime. It follows that, upon writing
$$A = \{10, 40, 70, 100\} \quad \text{and} \quad B = \{20, 30, 50, 60, 80, 90\},$$
either both $m+1$ and $n+1$ lie in B or exactly one lies in A and exactly one lies in B.

   This gives us a good way to count the answer by casing on m; if $m+1 \in A$, then check all values of $n+1$ lying in B, while if $m+1 \in B$, then check all values of $n+1$ lying in $A \cup B$ instead. Doing this and remembering to take into account that m and n are ordered yields the correct answer of $\boxed{52}$.

3. Points $A(0, 0)$ and $B(1, 1)$ are located on the parabola $y = x^2$. A third point C is positioned on this parabola between A and B such that $AC = CB = r$. What is r^2?

   Proposed by David Altizio

   Answer: $5 - 2\sqrt{5}$

   Solution. Let C be the point (t, t^2). Then by the Distance Formula the equality $AC = CB$ is equivalent to
$$t^2 + t^4 = r^2 = (1-t)^2 + (1-t^2)^2 = 2 - 2(t + t^2) + (t^2 + t^4).$$

Thus $t^2+t=1$. Solving yields $t=\frac{-1+\sqrt{5}}{2}$, so $t^2=\frac{1-\sqrt{5}}{2}$. This in turn implies

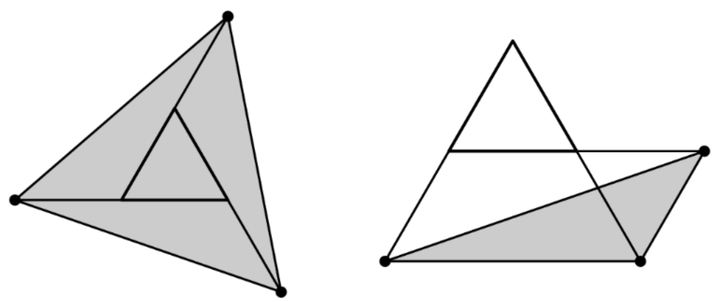
$$r^2=t^2+t^4=\frac{1-\sqrt{5}}{2}+\left(\frac{1-\sqrt{5}}{2}\right)^2=\boxed{5-2\sqrt{5}}.$$

4. Let $\triangle A_1B_1C_1$ be an equilateral triangle of area 60. Chloe constructs a new triangle $\triangle A_2B_2C_2$ as follows. First, she flips a coin. If it comes up heads, she constructs point A_2 such that B_1 is the midpoint of $\overline{A_2C_1}$. If it comes up tails, she instead constructs A_2 such that C_1 is the midpoint of $\overline{A_2B_1}$. She performs analogous operations on B_2 and C_2. What is the expected value of the area of $\triangle A_2B_2C_2$?

   Proposed by David Altizio

   Answer: 195

   Solution. Observe that there are two possible configurations which can arise. The left configuration occurs when either all coin flips are heads or all are tails, which happens with probability $\frac{1}{4}$. The right configuration occurs in the other case, which happens with probability $\frac{3}{4}$.



   Let $K=60$ denote the area of the original equilateral triangle for simplicity. The left triangle decomposes into the original triangle with area K and three congruent obtuse triangles with area $2K$, and so its total area is $7K$. The right triangle is congruent to one of these obtuse triangles and thus has area $2K$. It follows that the expected area of the shaded triangle is

$$2K\cdot\frac{3}{4}+7K\cdot\frac{1}{4}=\frac{13}{4}K=\boxed{195}.$$

5. On Misha's new phone, a passlock consists of six circles arranged in a 2×3 rectangle. The lock is opened by a continuous path connecting the six circles; the path cannot pass through a circle on the way between two others (e.g. the top left and right circles cannot be adjacent). For example, the left path shown below is allowed but the right path is not. (Paths are considered to be oriented, so that a path starting at A and ending at B is different from a path starting at B and ending at A. However, in the diagrams below, the paths are valid/invalid regardless of orientation.) How many passwords are there consisting of all six circles?

   Proposed by Max Aires and Ani Chowdhury

Solution. Number the circles in the top row $1, 2, 3$ and the circles in the bottom row $4, 5, 6$. The condition is equivalent to finding the number of permutations of $(1, 2, 3, 4, 5, 6)$ such that 1 and 3 are not adjacent and 4 and 6 are not adjacent.

We proceed using complementary counting and PIE. There are $6! = 720$ total permutations. The number of permutations with 1 and 3 next to each other is $5! \times 2 = 240$, and similarly the number of permutations with 4 and 6 next to each other is 240. Finally, the number of permutations with both adjacencies is $4! \times 2^2 = 96$. It follows that the requested answer is

$$720 - (240 + 240 - 96) = \boxed{336}.$$

6. Across all $x \in \mathbb{R}$, find the maximum value of the expression

$$\sin x + \sin 3x + \sin 5x.$$

Proposed by David Altizio

Answer: $\dfrac{72\sqrt{15}}{125}$

Solution. Write $\sin x + \sin 5x = 2 \sin 3x \cos 2x$ by the Sum to Product rules, so the expression to maximize becomes $\sin 3x(1 + 2 \cos 2x)$. Let $t = \sin x$. Note that

$$\sin 3x = 3t - 4t^3 = t(3 - 4t^2) = t(1 + 2 \cos 2x),$$

and so our expression miraculously becomes $t(3 - 4t^2)^2$. Now let this equal S; then by AM-GM

$$16S^2 = 16t^2 \left(3 - 4t^2\right)^2 \leq \left(\frac{16t^2 + 4(3 - 4t^2)}{5}\right)^2 = \left(\frac{12}{5}\right)^5,$$

and so $S \leq \frac{1}{4} \cdot (\frac{12}{5})^{5/2} = \boxed{\frac{72\sqrt{15}}{125}}$. Equality holds when $x = \arcsin\left(\frac{\sqrt{15}}{10}\right)$.

7. Suppose you start at 0, a friend starts at 6, and another friend starts at 8 on the number line. Every second, the leftmost person moves left with probability $\frac{1}{4}$, the middle person with probability $\frac{1}{3}$, and the rightmost person with probability $\frac{1}{2}$. If a person does not move left, they move right, and if two people are on the same spot, they are randomly assigned which one of the positions they are. Determine the expected time until you all meet in one point.

Proposed by Misha Ivkov

Answer: 16

Solution. We claim that S, the sum of pairwise distances between the three people, is a different random walk. Indeed, we can consider the change of the x-distance with each move:

Move	Probability	Change
LLL	1/24	0
LLR	1/24	4
LRL	1/12	0
LRR	1/12	4
RLL	1/8	-4
RLR	1/8	0
RRL	1/4	-4
RRR	1/4	0

We then rewrite our problem as a walker starting at $S = 4$ and trying to reach 0. The walker stays put with probability $\frac{1}{2}$, moves left 1 with probability $\frac{3}{8}$, and moves right with probability $\frac{1}{8}$. Let $\mathbb{E}[x, y]$ denote the time to get from x to y. We are interested in $\mathbb{E}[4, 0]$. Furthermore, note that the time the walker stops for is Geometric with probability $\frac{1}{2}$, so the expected stopping time is 2. Then we have the following recurrence for $\mathbb{E}[1, 0]$:

$$\mathbb{E}[1,0] = 2 + \frac{1}{4}\mathbb{E}[2,0] = 2 + \frac{1}{4}(\mathbb{E}[2,1] + \mathbb{E}[1,0]) = 2 + \frac{1}{2}\mathbb{E}[1,0]$$

implying $\mathbb{E}[1, 0] = 4$. We used linearity of expectation here and noted that $\mathbb{E}[x, y] = \mathbb{E}[x + 1, y + 1]$. From here, note that $\mathbb{E}[n, 0] = n\mathbb{E}[1, 0]$ so $\mathbb{E}[4, 0] = \boxed{16}$.

8. A positive integer n is *brgorable* if it is possible to arrange the numbers $1, 1, 2, 2, \ldots, n, n$ such that between any two k's there are exactly k numbers (for example, $n = 2$ is not brgorable, but $n = 3$ is as demonstrated by 3, 1, 2, 1, 3, 2). How many brgorable numbers are less than 2019?

Proposed by Tudor Popescu

Answer: 1008

Solution. We claim that the only brgorable numbers n are those for which $n \equiv 0$ or $3 \pmod 4$.

First we show that a number of form $4k+1$ or $4k+2$ is not brgorable. Color the numbers in black and white alternatively. Note that if a number is brgorable, then the positions of any two equal odd numbers must have the same color, while the positions of any two equal even ones must have different colors. Therefore, we must have an even number of odd numbers, so we have that n is of the form $4k$ or $4k + 3$.

It remains to find constructions for these numbers. First suppose $n = 4k$; then the construction is

$$\begin{aligned}&4k-4, 4k-6, \ldots, 2k, 4k-2, 2k-3, 2k-5, \ldots, 1, 4k-1, 1, 3, \ldots, 2k-3,\\ &\qquad 2k, 2k+2, \ldots, 4k-4, 4k, 4k-3, 4k-5, \ldots, 2k+1, 4k-2, 2k-2, 2k-4, \ldots,\\ &\qquad\qquad 2, 2k-1, 4k-1, 2, 4, \ldots, 2k-2, 2k+1, 2k+3, \ldots, 4k-3, 2k-1, 4k.\end{aligned}$$

For example, for $k = 3$ this yields the sequence

$$8,\ 6,\ 10,\ 3,\ 1,\ 11,\ 1,\ 3,\ 6,\ 8,\ 12,\ 9,\ 7,\ 10,\ 4,\ 2,\ 5,\ 11,\ 2,\ 4,\ 7,\ 9,\ 5,\ 12.$$

Now suppose $n = 4k - 1$. Then the construction is

$$\begin{aligned}&4k-4, 4k-6, \ldots, 2k, 4k-2, 2k-3, 2k-5, \ldots, 1, 4k-1, 1, 3, \ldots, 2k-3,\\ &\qquad 2k, 2k+2, \ldots, 4k-4, 2k-1, 4k-3, 4k-5, \ldots, 2k+1, 4k-2, 2k-2, 2k-4, \ldots, 2,\\ &\qquad\qquad 2k-1, 4k-1, 2, 4, \ldots, 2k-2, 2k+1, 2k+3, \ldots, 4k-3.\end{aligned}$$

For example, if $k = 2$, then the sequence is

$$4,\ 6,\ 1,\ 7,\ 1,\ 4,\ 3,\ 5,\ 6,\ 2,\ 3,\ 7,\ 2,\ 5.$$

Therefore, there are exactly 1008 brgorable numbers less than 2019.

9. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection satisfying $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$. Determine the minimum possible value of $f(n)/n$, taken over all possible f and all $n \le 2019$.

Proposed by Vijay Srinivasan

Answer: $\frac{2}{2017}$

Solution. First, it is clear that $f(1) = 1$ so $f(n) > 1$ for $n > 1$. Let $\mathcal{P}$ denote the set of primes. It is clear that for a bijection $g : \mathcal{P} \to \mathcal{P}$, there is a unique completely multiplicative bijection $f : \mathbb{N} \to \mathbb{N}$ with $f|_{\mathcal{P}} = g$. We claim for any f satisfying the conditions of the problem, there is such a bijection g with $f|_{\mathcal{P}} = g$. Suppose that for some prime p, $f(p)$ is not prime. Then there are $q, r > 1$ with $f(p) = qr$, so $p = f^{-1}(q)f^{-1}(r)$

expresses p as a product of integers > 1, a contradiction. Thus $f(\mathcal{P}) \subset \mathcal{P}$. If there is a prime p that does not appear in $f(\mathcal{P})$, then p also does not appear in $f(\mathbb{N})$, a contradiction since f is a bijection on the natural numbers. So $f(\mathcal{P}) \supset \mathcal{P}$ and so $f|_{\mathcal{P}}$ is a bijection as desired.

Then for prime numbers p we have $f(p) \geq 2$ and for composite numbers m we have $f(m) \geq 4$. Thus, for $1 \leq n \leq 2019$, we have $f(n)/n \geq \boxed{\frac{2}{2017}}$. This bound is achieved, for example, when $f(2) = 2017$, $f(2017) = 2$, and $f(p) = p$ for all $p \in \mathcal{P} \setminus \{2, 2017\}$.

10. Let $\triangle ABC$ be a triangle with side lengths a, b, and c. Circle ω_A is the A-excircle of $\triangle ABC$, defined as the circle tangent to BC and to the extensions of AB and AC past B and C respectively. Let $\mathcal{T}_A$ denote the triangle whose vertices are these three tangency points; denote $\mathcal{T}_B$ and $\mathcal{T}_C$ similarly. Suppose the areas of $\mathcal{T}_A$, $\mathcal{T}_B$, and $\mathcal{T}_C$ are 4, 5, and 6 respectively. Find the ratio $a : b : c$.

Proposed by David Altizio

Answer: $22 : 25 : 27$

Solution. Let the tangency points of ω_A with BC, AC, and AB be A', B', and C' respectively, and denote by I the incenter of $\triangle ABC$. Note that a simple angle chase yields $\angle A'B'C' = \angle BA'C' = \angle IBC$ and similarly $\angle A'C'B' = \angle ICB$, so $\triangle A'B'C' \sim \triangle IBC$. The ratio of similitude of these triangles is

$$\frac{2s\sin\frac{A}{2}}{a} = \frac{2s}{a}\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

As such the area T_A of $\mathcal{T}_A$ is

$$T_A = \frac{1}{2}ar \cdot \left(\frac{2s}{a}\sqrt{\frac{(s-b)(s-c)}{bc}}\right)^2 = \frac{2s^2r(s-b)(s-c)}{abc} = \frac{K^2}{2R(s-a)}.$$

This means that

$$K^2\left(\frac{1}{T_A} + \frac{1}{T_B}\right) = 2R[(s-a)+(s-b)] = 2Rc$$

and similar, so

$$\frac{a}{c} = \frac{2Ra}{2Rc} = \frac{\frac{1}{T_B} + \frac{1}{T_C}}{\frac{1}{T_A} + \frac{1}{T_B}} = \frac{T_A(T_B+T_C)}{T_C(T_B+T_A)}.$$

Plugging in the numbers yields $\frac{a}{c} = \frac{4\cdot 11}{6\cdot 9} = \frac{22}{27}$. Similarly, $\frac{b}{c} = \frac{25}{27}$. Therefore $a : b : c = \boxed{22 : 25 : 27}$.

11. Let S be a subset of the natural numbers such that $0 \in S$, and for all $n \in \mathbb{N}$, if n is in S, then both $2n+1$ and $3n+2$ are in S. What is the smallest number of elements S can have in the range $\{0, 1, \ldots, 2019\}$?

Proposed by Cody Johnson

Answer: 47

Solution. Set $N = 2019$ for simplicity. Let $f(n) = 2n+1$ and $3n+2$. The key observation is that

$$f(g(n)) = f(3n+2) = 6n+5 = g(2n+1) = f(g(n)),$$

i.e. that f and g commute.

With this in mind, let

$$\mathcal{S} = \{n \geq 0 : \exists\, p, q \geq 0 \mid n = f^p(g^q(0))\}.$$

Observe that any set S satisfying the properties in the problem statement must contain all elements of $\mathcal{S}$. Furthermore, by commutativity of f and g, $\mathcal{S}$ is closed under f and g. It follows that $\mathcal{S}$ is the smallest set satisyfing the problem constraints, where here “smallest” is referring to inclusion.

But now we may characterize all elements of $\mathcal{S}$ specifically, since

$$g^q(0) = 3^q - 1 \qquad \text{and} \qquad f^p(3^q - 1) = 2^p 3^q - 1.$$

It follows that an integer $0 \leq n \leq N$ is in $\mathcal{S}$ if and only if it can be written in the form $2^p 3^q - 1$ for some $p, q \geq 0$.

Finally, to compute the answer, note that all such numbers are distinct, so we may partition $\{0, 1, \ldots, N\} \cap \mathcal{S}$ based on the value of p. For a given p, write

$$2^p 3^q - 1 \leq 2019 \qquad \text{if and only if} \qquad 2^p \leq \frac{2020}{3^q}.$$

It follows that the number of elements of $\{0, 1, \ldots, N\} \cap \mathcal{S}$ in this form is $\lfloor \log_2(\frac{2020}{3^p}) \rfloor + 1$, and so the requested answer is

$$\sum_{m \geq 0} \left\lfloor \log_2 \left(\frac{2020}{3^m} \right) \right\rfloor + 1 = 11 + 10 + 8 + 7 + 5 + 4 + 2 = \boxed{47}.$$

12. Call a convex quadrilateral *angle-Pythagorean* if the degree measures of its angles are integers $w \leq x \leq y \leq z$ satisfying

$$w^2 + x^2 + y^2 = z^2.$$

Determine the maximum possible value of $x + y$ for an angle-Pythagorean quadrilateral.

Proposed by Gunmay Handa and Vijay Srinivasan

Answer: 207

Solution. Let $n = 180$. We have the constraints

$$w + x + y + z = 2n \qquad \text{and} \qquad w^2 + x^2 + y^2 = z^2.$$

Set $p = n - x$, $q = n - y$ for ease so that we have $w + z = p + q$ and hence also

$$(n - p)^2 + (n - q)^2 = (p + q)(z - w).$$

Since $z = p + q - w$ we have

$$p + q - 2w = \frac{(n - p)^2 + (n - q)^2}{p + q}$$

and so solving for w gives

$$w = n - \frac{n^2 - pq}{p + q}.$$

If we set $r = \frac{n^2 - pq}{p+q}$ then we want ordered pairs (p, q) for which r is an integer and $n - r = w < x = n - p$, i.e. $r > p$. We now write $p + q = S$. So we want to find the minimal S for which there exists a p such that

$$\frac{n^2 - p(S - p)}{S}$$

is an integer $> p$. An AM-GM bound on $n^2 - p(S - p)$ yields that $S \geq \lceil 2n(\sqrt{2} - 1) \rceil = 150$. We see that $S = 150$ gives solutions

$$(p, q) = (90, 60), (120, 30)$$

but in both of these cases we find that $w \leq 0$. For $S = 151$, we see that r being an integer is equivalent to $151 \mid (n^2 + p^2)$ which is impossible since 151 is a prime $\equiv 3 \pmod 4$. Similarly 152 can be eliminated since $19 \mid 152$. Having $S = 153$ finally gives a solution, namely $(w, x, y, z) = (4, 84, 123, 149)$. So $x + y$ is maximized at $360 - 153 = \boxed{207}$.

13. Points A, B, and C lie in the plane such that $AB = 13$, $BC = 14$, and $CA = 15$. A peculiar laser is fired from A perpendicular to $\overline{BC}$. After bouncing off BC, it travels in a direction perpendicular to CA. When it hits CA, it travels in a direction perpendicular to AB, and after hitting AB its new direction is perpendicular to BC again. If this process is continued indefinitely, the laser path will eventually approach some finite polygonal shape T_∞. What is the ratio of the perimeter of T_∞ to the perimeter of $\triangle ABC$?

Proposed by David Altizio

Answer: $\frac{168}{295}$

Solution. The shape T_∞ is actually $\triangle XYZ$, where $X \in AB$, $Y \in BC$, and $Z \in CA$ such that $ZY \perp BC$, $YX \perp AB$, and $XZ \perp CA$.

To prove this, for all positive integers n let $d_n = AX_n$, where X_n is the bouncing point of the laser on AB after n turns. By going around the triangle and using right-triangle trig to compute the locations of other bounce points, one sees that

$$d_{n+1} = c - \cos B(a - \cos C(b - d_n \cos A)) = M + Nd_n$$

for some universal constants M and N. Now because $|N| < 1$, the function $x \mapsto M + Nx$ is a contraction, and so by the Banach Fixed Point Theorem we see that the d_n converge to some fixed real number r. This proves the claim. (Banach is not necessary here; noting that $|d_{n+1} - d_n|$ decays geometrically is good enough too.)

We now propose three ways to finish.

- **Geometric Finish:** It is easy to see that $\triangle XYZ \sim \triangle ABC$ via an angle chase; for example, $\angle AYZ = 90^\circ - \angle AYB = \angle ABC$. These triangles are furthermore directly similar, and thus there exists a spiral similarity sending $\triangle XYZ$ to $\triangle ABC$. Let P denote the center of this spiral similarity. Then $\angle PZX = \angle PAB$, so quadrilateral $AXPZ$ is cyclic, which in turn implies $\angle PXZ = \angle PAZ = \angle PZY$. Repeating this argument cyclically yields $\angle PAB = \angle PBC = \angle PCA = \omega$, so in fact P is the first Brocard point of $\triangle ABC$ and ω the Brocard angle.

  To finish, remark that since Z is spirally sent to A under the spiral similarity, the ratio of the perimeters of the two triangles is $PZ : PA$. But note that since $AXPZ$ is cyclic, $\angle APZ = \angle AXZ = 90^\circ$, so this expression is actually equal to

$$\tan\omega = \frac{1}{\cot A + \cot B + \cot C} = \frac{4K}{a^2 + b^2 + c^2} = \boxed{\frac{168}{295}}.$$

- **Geometrico-Trigonometric Finish:** Let E be with $\overline{AE} \perp \overline{AB}$ and $\overline{DE} \perp \overline{AC}$, where D is the foot of A onto $\overline{BC}$. Observe that $Z \equiv \overline{BE} \cap \overline{AC}$, so since $\triangle YXZ \sim \triangle ADE \sim \triangle BCA$ we have that

$$\frac{YZ}{AB} = \frac{YZ/AE}{AB/AE} = \frac{BZ/BE}{BC/AD}.$$

  But

$$\frac{BZ}{ZE} = \frac{AB \sin\angle BAZ}{AE \sin\angle ZAE} = \frac{BC \sin A}{AD \cos A},$$

  whence

$$\frac{BZ/BE}{BC/AD} = \frac{\frac{BC \sin A}{BC \sin A + AD \cos A}}{\frac{BC}{AD}} = \frac{AD \sin A}{BC \sin A + AD \cos A}.$$

  Dividing through by $\sin A$, multiplying by BC in the numerator and denominator and using the fact that $AD = \frac{AB \cdot AC \cdot \sin A}{BC}$ yields

$$\frac{AD \sin A}{BC \sin A + AD \cos A} = \frac{AD \cdot BC}{BC^2 + AB \cdot AC \cos A} = \frac{4K}{a^2 + b^2 + c^2}$$

  by the Law of Cosines, as desired.

- **Trigonometric Finish:** The real number r is the unique fixed point of f, i.e. the solution to $r = M+Nr$. The solution to this is $r = \frac{M}{1-N}$, and so, after deducing $\triangle XYZ \sim \triangle ABC$ as in the first solution, the desired ratio is

$$\begin{aligned}\frac{XZ}{BC} = \frac{r\sin A}{b} &= \frac{(c - a\cos B + b\cos B\cos C)\sin A}{(1+\cos A\cos B\cos C)b} = \frac{(b\cos A + b\cos B\cos C)\sin A}{(1+\cos A\cos B\cos C)b} \\ &= \frac{\sin A\sin B\sin C}{1+\cos A\cos B\cos C} \overset{(*)}{=} \frac{K}{2R^2(1+\cos A\cos B\cos C)} \overset{(**)}{=} \frac{K}{R^2(\sin^2 A+\sin^2 B+\sin^2 C)} \\ &= \frac{4K}{a^2+b^2+c^2},\end{aligned}$$

and so we have the same answer as above. Here $(*)$ is due to the identity $K = 2R^2\sin A\sin B\sin C$ while $(**)$ is due to the identity $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$.

14. Consider the following function:

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procedure M(x)
  if $0 \le x \le 1$ **then**
    return x
  return M($x^2 \bmod 2^{32}$)

Let $f : \mathbb{N} \to \mathbb{N}$ be defined such that $f(x) = 0$ if M(x) does not terminate, and otherwise $f(x)$ equals the number of calls made to M during the running of M(x), not including the initial call. For example, $f(1) = 0$ and $f(2^{31}) = 1$. Compute the number of ones in the binary expansion of

$$f(0) + f(1) + f(2) + \cdots + f(2^{32} - 1).$$

Proposed by Misha Ivkov and Theodore Li

Answer: 15

Solution. Note first that all numbers terminate and that the algorithm just returns if a number is even or odd. So, let's do those two cases separately.

First, we claim that the order of 3 modulo 2^k is 2^{k-2}. Note that $v_2(j!) \le 2j-3$ if and only if $j > 1$. Hence $4^j\binom{2^k-3}{j} \equiv 0 \bmod 2^k$ iff $j > 1$. Then $3^{2^{k-3}} = (4-1)^{2^{k-3}} \equiv -4\times 2^{k-3}+1 \bmod 2^k$, so $\text{ord}_{2^k}(3) > 2^{k-3}$. However, we can adapt the same argument and find that $3^{2^{k-2}} \equiv -4\times 2^{k-2}+1 \equiv 1 \bmod 2^k$ so $\text{ord}_{2^k}(3) = 2^{k-2}$.

Define $S = \{3^i \bmod 2^k \mid 0 \le i < 2^{k-2}\}$. Now note that at least one of $2^k-1, 2^{k-1}-1, 2^{k-1}+1$ is not in S since they all square to 1. Let m be one of the ones which is not in S. Then note that $f(3^i) = f(m\times 3^i)$ except when $i = 0$.

Further, note that $f(1) = 0$ and $f(m) = 1$. Now let's get an exact value for $f(3^a)$. Note that $f(3) = 30$ (since we go $3^1, 3^2, \ldots, 3^{2^i}, \ldots, 3^{2^{30}}$ for a total of 30 iterations). Therefore, we can deduce that $f(3^{2^a b}) = 30 - a$ for all odd b. Finally the sum we wish to find is $1 + 2\sum_{x=1}^{2^{30}-1} f(3^x)$. There are 2^{29} such odd x, 2^{28} such x which are not divisible by 4, and so on. So in fact

$$1 + 2\sum_{x=1}^{2^{30}-1} f(3^x) = 1 + \sum_{a=1}^{30} a2^a = 3 + 29\times 2^{31}$$

The even case happens to be much easier. Let $x = 2^a b$ be even. Then we double the exponent at every iteration. For example, if $a = 1$, then we will go $2^{2^0}b, 2^{2^1}b^2, 2^{2^2}b^4, \ldots, 2^{2^5}b^{32}$ which means $f(2b) = 5$ for all odd b. Similarly, we can compute that $f(4b) = f(8b) = 4$. Continuing onward, we must have that $f(16b) = \ldots = f(128b) = 3$. In general, there are 2^{30} x such that $f(x) = 5$, $(2^2-1)\times 2^{28}$ with $f(x) = 4$,

$(2^4-1)\times 2^{24}$ with $f(x)=3$, $(2^8-1)\times 2^{16}$ with $f(x)=2$, and $2^{16}-1$ with $f(x)=1$. Hence the total even sum is

$$5\times 2^{30}+12\times 2^{28}+45\times 2^{24}+510\times 2^{16}+2^{16}-1$$

Therefore our number is

$$\begin{aligned} n &= 29\times 2^{31}+5\times 2^{30}+12\times 2^{28}+45\times 2^{24}+511\times 2^{16}+2 \\ &= 33\times 2^{31}+45\times 2^{24}+511\times 2^{16}+2 \\ &= 2^{36}+2^{31}+(2^5+2^3+2^2+2^0)\times 2^{24}+(2^8+2^8-1)\times 2^{16}+2 \\ &= 2^{36}+2^{31}+2^{29}+2^{27}+2^{26}+2^{25}+2^{23}+\ldots+2^{16}+2^1 \end{aligned}$$

which has $\boxed{15}$ ones:

$$1000010101110111111110000000000000010.$$

15. Call a polynomial P *prime covering* if for every prime p, there exists an integer n for which p divides $P(n)$. Determine the number of ordered triples of integers (a,b,c), with $1\le a<b<c\le 25$, for which $P(x)=(x^2-a)(x^2-b)(x^2-c)$ is prime-covering.

Proposed by Vijay Srinivasan

Answer: 1194

Solution. We claim that the result holds iff at least one of a, b, c, or abc is a perfect square.

First we show that this condition works. If a is a perfect square, then setting $n:=\sqrt{a}\in\mathbb{N}$ gives $P(n)=0$, and so in particular $p\mid P(n)$ for all positive integers n. Analogous reasoning works when b and c are squares. Now suppose abc is a perfect square, and note that for any prime p

$$1=\left(\frac{abc}{p}\right)=\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)\left(\frac{c}{p}\right).$$

Thus it is impossible for a, b, and c to simultaneously not be quadratic residues modulo p, meaning there must exist some n for which $p\mid P(n)$.

The reverse direction is significantly trickier, and crucially makes use of the following lemma.

Lemma 1. *Let $p_1,p_2,\cdots,p_n$ be primes and $e_1,e_2,\cdots,e_n\in\{-1,1\}$. Then there exists a prime q with the property that $(\frac{p_i}{q})=e_i$ for all $1\le i\le n$.*

Proof. For each $1\le i\le n$, let α_i be some (nonzero) quadratic residue modulo p_i if $e_i=1$ and let α_i be some nonzero nonquadratic residue modulo p_i if $e_i=-1$. Furthermore, set $\alpha=5$ iff there exists some i with $p_i=2$ and $e_i=-1$, and let $\alpha=1$ otherwise.

Now consider the system of equations

$$\begin{cases} x &\equiv \alpha \pmod 8, \\ x &\equiv \alpha_1 \pmod{p_i}, \\ &\vdots \\ x &\equiv \alpha_n \pmod{p_n}. \end{cases}$$

Note that by CRT this system has a unique solution

$$x\equiv N \pmod{8p_1\ldots p_n}$$

(and in particular is not inconsistent if $p_i=2$ for some i). Furthermore, $\gcd(N,8p_1\ldots p_n)=1$ since $\gcd(\alpha_i,p_i)=1$ for all i. It follows by Dirichlet's Theorem that there exists some prime q satisfying this system of congruences. We claim that this is the prime q we seek.

To prove this, let i be arbitrary. We case on the value of p_i.

- If $p_i = 2$, then recall
$$\left(\frac{2}{q}\right) = \begin{cases} 1 & \text{if } q \equiv 1, 7 \pmod 8, \\ -1 & \text{if } q \equiv 3, 5 \pmod 8. \end{cases}$$
But $q \equiv \alpha \pmod 8$, where α was 1 if $e_i = 1$ and α was 5 if $e_i = -1$. It follows that $(\frac{2}{p_i}) = e_i$.
- Now suppose $p_i \geq 3$ is an odd prime. Note that $q \equiv 1 \pmod 4$, so by the Law of Quadratic Reciprocity,
$$\left(\frac{p_i}{q}\right)\left(\frac{q}{p_i}\right) = (-1)^{(p_i-1)(q-1)/4} = 1,$$
whence $(\frac{p_i}{q}) = (\frac{q}{p_i})$. In particular, $(\frac{p_i}{q}) = 1$ iff q is a quadratic residue modulo p_i, which is exactly what the condition $x \equiv \alpha_i \pmod{p_i}$ forces.

We are done. $\square$

We now proceed with the proof. Suppose that none of a, b, and c are squares. Then WLOG assume they are squarefree since in general for integers m and n and any prime p, it is true that $(\frac{mn^2}{p}) = (\frac{m}{p})$. Suppose abc is not a square. Let $\{p_1, p_2, \cdots, p_n\}$ be the set of primes dividing abc. Since abc is not a square, it follows that one of these primes - WLOG let it be p_1 - divides exactly 1 or exactly 3 elements of the set $\{a, b, c\}$. If p_1 divides all of a, b, and c, defining

$$e_1 = -1 \qquad \text{and} \qquad e_2 = e_3 = \cdots = e_n = 1$$

guarantees the existence of a prime q for which $(\frac{a}{q}) = (\frac{b}{q}) = (\frac{c}{q}) = -1$, and hence contradicts the assumption that $\{P(n)\}$ contains a multiple of q. So p_1 must divide exactly one of a, b, c; again WLOG suppose $p_1 \mid a$. If b and c share a prime factor (say p_2), then define $e_2 = -1$ and $e_3 = e_4 = \cdots = e_n = 1$. If b and c are coprime, WLOG suppose that $p_2 \mid b$ and $p_3 \mid c$. Set

$$e_2 = e_3 = -1 \qquad \text{and} \qquad e_4 = e_5 = \cdots = e_n = 1.$$

We can write $a = p_1 p_{m_1} p_{m_2} \cdots p_{m_r}$, and defining $e_1 = -e_{m_1} e_{m_2} \cdots e_{m_r}$ reveals that we can again find a prime q which contradicts the assumption of the problem. So if abc is not a square, there is some prime q for which $\{P(n)\}$ contains no multiples of q.

Now we proceed with the counting. We split into cases.

- The number of sets $\{a, b, c\}$ which contain at least one perfect square is, by complementary counting, equal to $\binom{25}{3} - \binom{20}{3} = 1160$.
- Now suppose a, b, and c are all squarefree. The prime divisors of abc must come from the set $\{2, 3, 5, 7, 11\}$; else at least two of a, b, or c must be divisible by a prime which is at least 13. Furthermore, all products of one or two primes from this set lie in the set $\{1, 2, \ldots, 25\}$. Hence the set $\{a, b, c\}$ is either of the form $\{p, q, pq\}$ or $\{pq, pr, qr\}$ for some distinct primes p, q, r; the number of such sets is thus $\binom{5}{2} + \binom{5}{3} = 20$.
- Finally, suppose a, b, c are all not perfect squares but at least one is not squarefree. The only integers which are neither perfect squares nor squarefree with prime divisors in the set $\{2, 3, 5, 7, 11\}$ are 2^3, $2^2 \times 3$, $2^2 \times 5$, and 2×3^2. A quick count yields 14 additional sets, namely
$$\{1,2,8\}, \{1,8,18\}, \{3,6,8\}, \{3,8,24\}, \{5,8,10\}, \{6,8,12\}, \{7,8,14\},$$
$$\{8,10,20\}, \{8,11,22\}, \{8,12,24\}, \{2,6,12\}, \{2,12,24\}, \{5,12,15\}, \text{ and } \{7,12,21\}.$$

It follows that the requested answer is $1160 + 34 = \boxed{1194}$.