CMIMC 2019

Team Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 15 short-answer problems to be solved in 60 minutes.
3. Write your team name and team ID on your answer sheet.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. All answers are integers.
7. If you believe that the test contains an error, submit your protest in writing to Doherty Hall 2302.

Team

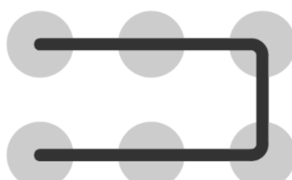
1. David recently bought a large supply of letter tiles. One day he arrives back to his dorm to find that some of the tiles have been arranged to read CENTRAL MICHIGAN UNIVERSITY. What is the smallest number of tiles David must remove and/or replace so that he can rearrange them to read CARNEGIE MELLON UNIVERSITY?

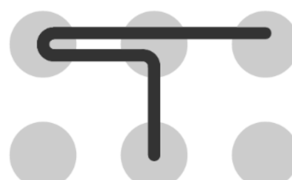
2. Determine the number of ordered pairs of positive integers (m, n) with $1 \le m \le 100$ and $1 \le n \le 100$ such that
$$\gcd(m+1, n+1) = 10\gcd(m, n).$$

3. Points $A(0,0)$ and $B(1,1)$ are located on the parabola $y = x^2$. A third point C is positioned on this parabola between A and B such that $AC = CB = r$. What is r^2?

4. Let $\triangle A_1B_1C_1$ be an equilateral triangle of area 60. Chloe constructs a new triangle $\triangle A_2B_2C_2$ as follows. First, she flips a coin. If it comes up heads, she constructs point A_2 such that B_1 is the midpoint of $\overline{A_2C_1}$. If it comes up tails, she instead constructs A_2 such that C_1 is the midpoint of $\overline{A_2B_1}$. She performs analogous operations on B_2 and C_2. What is the expected value of the area of $\triangle A_2B_2C_2$?

5. On Misha's new phone, a passlock consists of six circles arranged in a 2×3 rectangle. The lock is opened by a continuous path connecting the six circles; the path cannot pass through a circle on the way between two others (e.g. the top left and right circles cannot be adjacent). For example, the left path shown below is allowed but the right path is not. (Paths are considered to be oriented, so that a path starting at A and ending at B is different from a path starting at B and ending at A. However, in the diagrams below, the paths are valid/invalid regardless of orientation.) How many passlocks are there consisting of all six circles?





6. Across all $x \in \mathbb{R}$, find the maximum value of the expression
$$\sin x + \sin 3x + \sin 5x.$$

7. Suppose you start at 0, a friend starts at 6, and another friend starts at 8 on the number line. Every second, the leftmost person moves left with probability $\frac{1}{4}$, the middle person with probability $\frac{1}{3}$, and the rightmost person with probability $\frac{1}{2}$. If a person does not move left, they move right, and if two people are on the same spot, they are randomly assigned which one of the positions they are. Determine the expected time until you all meet in one point.

8. A positive integer n is *brgorable* if it is possible to arrange the numbers $1, 1, 2, 2, ..., n, n$ such that between any two k's there are exactly k numbers (for example, $n = 2$ is not brgorable, but $n = 3$ is as demonstrated by 3, 1, 2, 1, 3, 2). How many brgorable numbers are less than 2019?

9. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection satisfying $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$. Determine the minimum possible value of $f(n)/n$, taken over all possible f and all $n \le 2019$.

10. Let $\triangle ABC$ be a triangle with side lengths a, b, and c. Circle ω_A is the A-excircle of $\triangle ABC$, defined as the circle tangent to BC and to the extensions of AB and AC past B and C respectively. Let $\mathcal{T}_A$ denote the triangle whose vertices are these three tangency points; denote $\mathcal{T}_B$ and $\mathcal{T}_C$ similarly. Suppose the areas of $\mathcal{T}_A$, $\mathcal{T}_B$, and $\mathcal{T}_C$ are 4, 5, and 6 respectively. Find the ratio $a : b : c$.

11. Let S be a subset of the natural numbers such that $0 \in S$, and for all $n \in \mathbb{N}$, if n is in S, then both $2n + 1$ and $3n + 2$ are in S. What is the smallest number of elements S can have in the range $\{0, 1, \ldots, 2019\}$?

12. Call a convex quadrilateral *angle-Pythagorean* if the degree measures of its angles are integers $w \le x \le y \le z$ satisfying
$$w^2 + x^2 + y^2 = z^2.$$
Determine the maximum possible value of $x + y$ for an angle-Pythagorean quadrilateral.

13. Points A, B, and C lie in the plane such that $AB = 13$, $BC = 14$, and $CA = 15$. A peculiar laser is fired from A perpendicular to $\overline{BC}$. After bouncing off BC, it travels in a direction perpendicular to CA. When it hits CA, it travels in a direction perpendicular to AB, and after hitting AB its new direction is perpendicular to BC again. If this process is continued indefinitely, the laser path will eventually approach some finite polygonal shape T_∞. What is the ratio of the perimeter of T_∞ to the perimeter of $\triangle ABC$?

14. Consider the following function.

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procedure M(x)
    if $0 \le x \le 1$ **then**
        return x
    return M($x^2 \bmod 2^{32}$)

Let $f : \mathbb{N} \to \mathbb{N}$ be defined such that $f(x) = 0$ if M(x) does not terminate, and otherwise $f(x)$ equals the number of calls made to M during the running of M(x), not including the initial call. For example, $f(1) = 0$ and $f(2^{31}) = 1$. Compute the number of ones in the binary expansion of
$$f(0) + f(1) + f(2) + \cdots + f(2^{32} - 1).$$

15. Call a polynomial P *prime covering* if for every prime p, there exists an integer n for which p divides $P(n)$. Determine the number of ordered triples of integers (a, b, c), with $1 \le a < b < c \le 25$, for which $P(x) = (x^2 - a)(x^2 - b)(x^2 - c)$ is prime-covering.