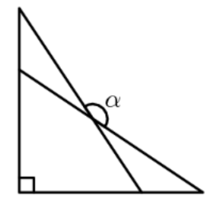
Geometry Solutions

1. The figure to the right depicts two congruent triangles with angle measures 40°, 50°, and 90°. What is the measure of the obtuse angle α formed by the hypotenuses of these two triangles?



   Proposed by David Altizio

   Answer: 170°

   Solution. The intersection of the two triangles is a convex quadrilateral with angle measures 90°, 50°, 50°, and α, whence $\alpha = 360° - 90° - 2 \cdot 50° = \boxed{170°}$.

2. Suppose X, Y, Z are collinear points in that order such that $XY = 1$ and $YZ = 3$. Let W be a point such that $YW = 5$, and define O_1 and O_2 as the circumcenters of triangles $\triangle WXY$ and $\triangle WYZ$, respectively. What is the minimum possible length of segment $\overline{O_1O_2}$?

   Proposed by Gunmay Handa

   Answer: 2

   Solution. Let P_1 and P_2 be the projections of O_1 and O_2 respectively onto $\overline{XZ}$. Then P_1 is the midpoint of $\overline{XY}$ and P_2 is the midpoint of $\overline{YZ}$, and so

$$O_1O_2 \geq P_1P_2 = \frac{1}{2}XZ = \boxed{2}.$$

3. Let ABC be an equilateral triangle with side length 2, and let M be the midpoint of $\overline{BC}$. Points X and Y are placed on AB and AC respectively such that $\triangle XMY$ is an isosceles right triangle with a right angle at M. What is the length of $\overline{XY}$?

   Proposed by David Altizio

   Answer: $3 - \sqrt{3}$

   Solution. Let P and Q be the feet of the perpendiculars from X and Y to BC. As $\triangle MXY$ is a $45-45-90$ triangle, it follows that $XYQP$ is a rectangle with $XY = 2YQ = 2XP$. With this in mind, let $s = XY$; then $PB = \frac{1}{\sqrt{3}}PX = \frac{s}{2\sqrt{3}}$, and similarly $QC = \frac{s}{2\sqrt{3}}$. Thus

$$2 = BC = BP + PQ + QC = \frac{s}{\sqrt{3}} + s,$$

   and so $s = \frac{2}{1+\frac{1}{\sqrt{3}}} = \boxed{3 - \sqrt{3}}$.

4. Suppose $\mathcal{T} = A_0A_1A_2A_3$ is a tetrahedron with $\angle A_1A_0A_2 = \angle A_2A_0A_3 = \angle A_3A_0A_1 = 90°$, $A_0A_1 = 5, A_0A_2 = 12$ and $A_0A_3 = 9$. A cube $A_0B_0C_0D_0E_0F_0G_0H_0$ with side length s is inscribed inside $\mathcal{T}$ with $B_0 \in \overline{A_0A_1}, D_0 \in \overline{A_0A_2}, E_0 \in \overline{A_0A_3}$, and $G_0 \in \triangle A_1A_2A_3$; what is s?

   Proposed by Gunmay Handa

   Answer: $\frac{180}{71}$

   Solution. Let $A_0 = (0,0,0)$, $A_1 = (5,0,0)$, $A_2 = (0,12,0)$, $A_3 = (0,0,9)$. The equation of the plane containing $\triangle A_1A_2A_3$ is $\frac{x}{5} + \frac{y}{12} + \frac{z}{9} = 1$, and this plane must contain the point (s,s,s), so

$$s = \frac{1}{\frac{1}{5} + \frac{1}{12} + \frac{1}{9}} = \boxed{\frac{180}{71}}.$$

5. Let $MATH$ be a trapezoid with $MA = AT = TH = 5$ and $MH = 11$. Point S is the orthocenter of $\triangle ATH$. Compute the area of quadrilateral $MASH$.

Proposed by David Altizio

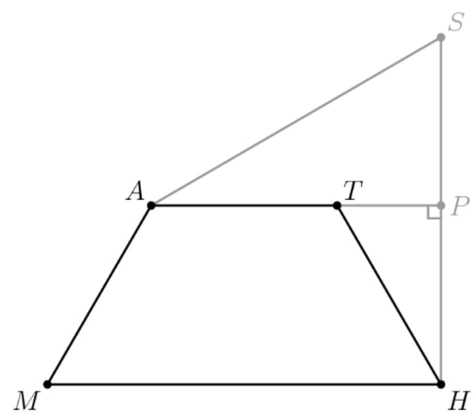
Answer: 62

Solution. Let P be the intersection point of AT and SH. Since S is the orthocenter of $\triangle ATH$, $SH \perp AT$, so P is the foot of the perpendicular from T to SH. Since $MATH$ is an isosceles trapezoid, $TP = \frac{11-5}{2} = 3$, so by Pythagorean Theorem $PH = 4$. Now a little angle chasing gives

$$\angle SAP = 90^\circ - \angle ASH = \angle THP,$$

so $\triangle ASP \sim \triangle HTP$, which in turn implies $\frac{SP}{PA} = \frac{TP}{PH}$, or $SP = \frac{8\cdot 3}{4} = 6$. It remains to compute

$$[MASH] = [MAPH] + [ASP] = \tfrac{1}{2}\cdot 4(8+11) + \tfrac{1}{2}\cdot 8\cdot 6 = 38 + 24 = \boxed{62}.$$



6. Let ABC be a triangle with $AB = 209$, $AC = 243$, and $\angle BAC = 60^\circ$, and denote by N the midpoint of the major arc $\widehat{BAC}$ of circle $\odot(ABC)$. Suppose the parallel to AB through N intersects $\overline{BC}$ at a point X. Compute the ratio $\frac{BX}{XC}$.

Proposed by David Altizio

Answer: $\frac{34}{209}$

Solution. Note that $\angle BNC = \angle BAC = 60^\circ$, so $\triangle NBC$ is equilateral. It follows by Ptolemy that $AN = CN - BN = 34$.

Now let AX intersect $\odot(ABC)$ again at P. Since $AB \parallel PN$, $ANPB$ is an isosceles trapezoid, so $BP = AN = 34$. Furthermore, since $AP = BN = BC$, quadrilateral $ABPC$ is also an isosceles trapezoid, meaning that $PC = AB = 209$. It follows by Angle Bisector that

$$\frac{BX}{XC} = \frac{BP}{PC} = \boxed{\frac{34}{209}}.$$

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Denote by ω its incircle. A line ℓ tangent to ω intersects $\overline{AB}$ and $\overline{AC}$ at X and Y respectively. Suppose $XY = 5$. Compute the positive difference between the lengths of $\overline{AX}$ and $\overline{AY}$.

Proposed by David Altizio

Solution. First compute the radius r of ω to be 4. Remark that since $BXYC$ is a circumscribed quadrilateral, Pitot's Theorem yields $BX + YC = 14 + 5 = 19$, so $AX + AY = (13 + 15) - 19 = 9$. This gives that the semiperimeter s of $\triangle AXY$ is equal to 7. In turn, since ω is the A-excircle of $\triangle AXY$, we have

$$[AXY] = r(s - XY) = 4(7 - 5) = 8.$$

As a result,

$$\frac{AX \cdot AY}{AB \cdot AC} = \frac{[AXY]}{[ABC]} = \frac{8}{84} = \frac{2}{21},$$

so $AX \cdot AY = \frac{2}{21} \cdot 13 \cdot 15 = \frac{130}{7}$. Finally, remark that

$$(AX - AY)^2 = (AX + AY)^2 - 4AX \cdot AY = 81 - 4 \cdot \tfrac{130}{7} = \tfrac{47}{7},$$

and so the requested answer is $\boxed{\sqrt{\frac{47}{7}}}$.

8. Consider the following three lines in the Cartesian plane:

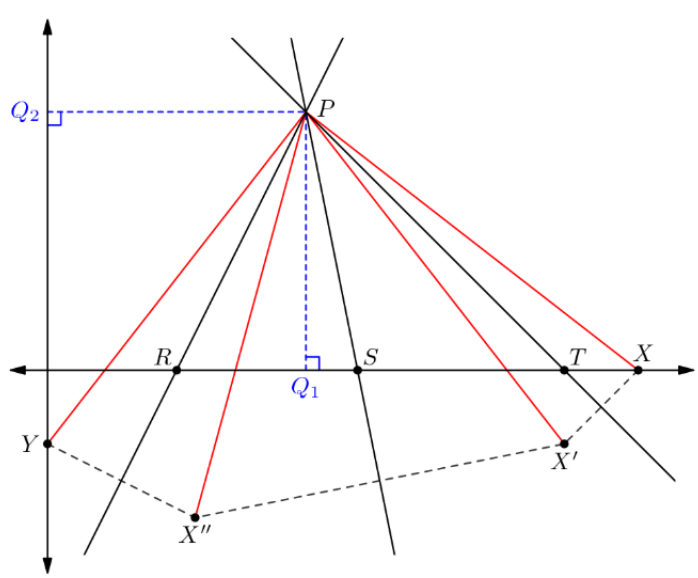
$$\begin{cases} \ell_1 : & 2x - y = 7 \\ \ell_2 : & 5x + y = 42 \\ \ell_3 : & x + y = 14 \end{cases}$$

and let $f_i(P)$ correspond to the reflection of the point P across ℓ_i. Suppose X and Y are points on the x and y axes, respectively, such that $f_1(f_2(f_3(X))) = Y$. Let t be the length of segment XY; what is the sum of all possible values of t^2?

Proposed by Gunmay Handa

Answer: 260

Solution. Observe that the composition of three reflections is also a reflection, and hence X and Y are unique. A bit of experimentation reveals that the diagram must look like the one below; through the rest of the solution, we will use the labels found there.



Note that since $PQ_1 = PQ_2 = 7$ and $PX = PY$, $\triangle PQ_2Y \cong \triangle PQ_1X$, so $\angle YPX = 90°$. Let $\angle YPR = \angle RPX'' = \alpha$, $\angle X''PS = \angle X'PS = \beta$, and $\angle XPT = \angle X'PT = \gamma$. From the previous perpendicularity,

$\alpha + \beta + \gamma = 45°$. But $\angle PTS = 45°$ as well, so $\angle Q_1PS = 45° - (\beta + \gamma) = \alpha$. A quick computation gives $SQ_1 = \frac{42}{5} - 7 = \frac{7}{5}$, and so

$$\tan \angle Q_2PY = \tan(\angle Q_2PR - \angle RPY) = \frac{2 - \frac{1}{5}}{1 + \frac{2}{5}} = \frac{9}{7}.$$

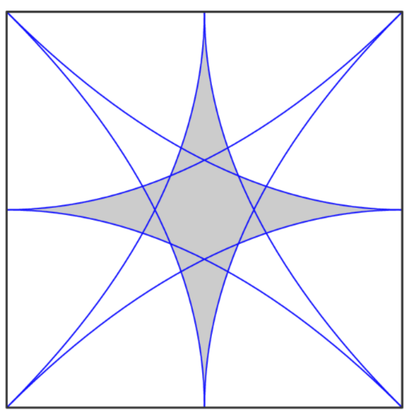
Hence $Q_2Y = 9$, and so $YX^2 = 2PY^2 = 2(7^2 + 9^2) = \boxed{260}$.

9. Let $ABCD$ be a square of side length 1, and let P_1, P_2 and P_3 be points on the perimeter such that $\angle P_1P_2P_3 = 90°$ and P_1, P_2, P_3 lie on different sides of the square. As these points vary, the locus of the circumcenter of $\triangle P_1P_2P_3$ is a region $\mathcal{R}$; what is the area of $\mathcal{R}$?

Proposed by Gunmay Handa

Answer: $\frac{23-16\sqrt{2}}{3}$

Solution. Suppose $P_1 \in \overline{AB}$, $P_3 \in \overline{BC}$ and $P_2 \in \overline{CD}$. Then $B \in \odot(P_1P_2P_3)$ whose circumcenter M is the midpoint of $\overline{P_1P_3}$, so that $MB = MP_2 \leq \text{dist}(M, \overline{CD})$. Hence, as P_1, P_2, P_3 vary across these desired segments, we see that M is bounded by the region of points that is equidistant from B and CD; i.e. the parabola with focus B and directrix CD. For all possible P_1, P_2, P_3 as the problem dictates, we obtain a region bounded by 8 parabolas: for each vertex, we take the two parabolas whose foci are each that vertex and whose directrices are the sides that do not contain the chosen vertex, as shown below.



Scale by a factor of 2 so that $A = (-1,-1), B = (1,-1), C = (1,1)$ and $D = (-1,1)$, and let ℓ_{CD}, ℓ_{BC} be the perpendicular bisectors of segments $\overline{CD}$ and $\overline{BC}$, respectively. Let $\mathcal{P}_{C,AB}$ be the parabola with focus C and directrix AB, and define $\mathcal{P}_{C,AD}$ analogously; observe that the area of $\mathcal{R}$ is 4 times the region $\mathcal{R}_1$ bounded by $\ell_{CD}, \ell_{BC}, \mathcal{P}_{C,AB}$ and $\mathcal{P}_{C,AD}$. We can compute the equations of $\mathcal{P}_{C,AB}$ as $y = \left(\frac{x-1}{2}\right)^2$ and $\mathcal{P}_{C,AD}$ as $x = \left(\frac{y-1}{2}\right)^2$, so $X \equiv \mathcal{P}_{C,AD} \cap \mathcal{P}_{C,AB} = (3 - 2\sqrt{2}, 3 - 2\sqrt{2})$ (where X necessarily lies in the interior of $ABCD$). Then

$$|\mathcal{R}_1| = (3 - 2\sqrt{2})^2 + 2\int_{3-2\sqrt{2}}^{1} \left(\frac{x-1}{2}\right)^2 dx$$
$$= 17 - 12\sqrt{2} + 2\left[\frac{(x-1)^3}{12}\right]\Bigg|_{3-2\sqrt{2}}^{1} = \frac{23 - 16\sqrt{2}}{3}.$$

Remembering to scale down by 4, our final answer is

$$4 \cdot \frac{1}{4} \cdot \frac{23 - 16\sqrt{2}}{3} = \boxed{\frac{23 - 16\sqrt{2}}{3}}.$$

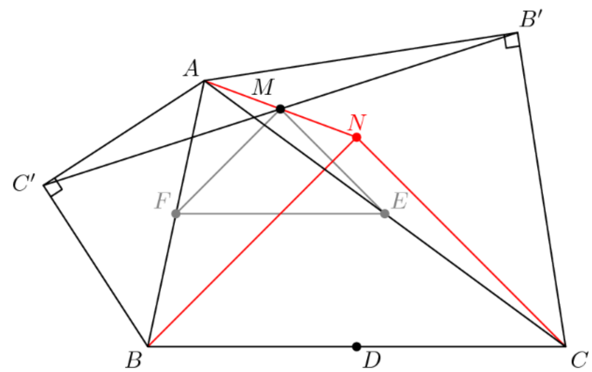
10. Suppose ABC is a triangle, and define B_1 and C_1 such that $\triangle AB_1C$ and $\triangle AC_1B$ are isosceles right triangles on the exterior of $\triangle ABC$ with right angles at B_1 and C_1, respectively. Let M be the midpoint of $\overline{B_1C_1}$; if $B_1C_1 = 12$, $BM = 7$ and $CM = 11$, what is the area of $\triangle ABC$?

Proposed by Gunmay Handa

Answer: $24\sqrt{10} - 49$

Solution. Define A_1 analogously to B_1 and C_1; we first claim that $AA_1 = B_1C_1$. Indeed, define C' to be the rotation of B around A $90°$ clockwise, so that $\triangle AB_1C_1 \sim \triangle ACC'$ and $\triangle ABA_1 \sim \triangle C'BC$. Moreover, the ratio of similitude between these two pairs of triangles is equal, and so $AA_1/C'C = B_1C_1/C'C$, which yields the desired equality.

Let D, E, and F be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively, and set N to be the reflection of A_1 about BC. Consider the spiral similarity sending $\triangle C_1BA$ to $\triangle B_1AC$, and note that $\triangle MFE$ is the triangle which results halfway through this spiral similarity; thus by the Gliding Principle $\triangle MFE$ is also an isosceles right triangle. It follows that the homothety centered at A with scale factor 2 sends $\triangle MFE$ to $\triangle NBC$, and so M is the midpoint of $\overline{AN}$. This in turn implies $MD = \frac{1}{2}AA_1 = 6$.



Applying the median length formula to $\triangle BMC$ yields $2(7^2+11^2) = 12^2+BC^2$, so $BC = 14$; Heron's Formula thus tells us that the area of $\triangle BMC$ is $12\sqrt{10}$. As a result, since M is the midpoint of $\overline{AN}$,

$$[BAC] = 2[BMC] - [BNC] = 2 \cdot 12\sqrt{10} - \tfrac{1}{2} \cdot 14 \cdot 7 = \boxed{24\sqrt{10} - 49}.$$