Combinatorics and Computer Science

1. Patrick tosses four four-sided dice, each numbered one through four. What's the probability that their product is a multiple of four?

   Proposed by Patrick Lin

   Answer: $\frac{13}{16}$

   Solution. Instead we can count the probability that the product is not a multiple of four. This means that either there are no 2/4s or exactly one 2. The number of ways for this is $2^4 + 4 \cdot 2^3 = 48$. Hence the probability that the product is a multiple of four is $1 - \frac{48}{256} = \frac{13}{16}$.

2. How many ways are there to color the vertices of a cube red, blue, or green such that no edge connects two vertices of the same color? Rotations and reflections are considered distinct colorings.

   Proposed by Patrick Lin

   Answer: 114

   Solution. Call the colors R, G, and B, and for the purposes of this solution identify the vertices of the cube as 000, 001, ..., 111. Assume without loss of generality that 000 is R, then there are eight ways to color its neighbors. In six of these, we may assume that 001 is G and both 010 and 100 are B. Then 011 and 101 are R, which leaves three possibilities for 110 and 111 together, and $3 \cdot 6 \cdot 3 = 54$ colorings in this case.

   In the other two cases, all of the neighbors of 000 are G. If 111 is also G, then each of 110, 101, 011 may be either R or B, giving eight colorings. If 111 is not G, then 110, 101, and 011 are fixed by 111, giving two. This case thus yields $3 \cdot 2 \cdot 10 = 60$ colorings. So the answer is $54 + 60 = \boxed{114}$.

3. How many ordered triples (a, b, c) of integers with $1 \le a \le b \le c \le 60$ satisfy $a \cdot b = c$?

   Proposed by Misha Ivkov

   Answer: 134

   Solution. Fix c. If c is not a perfect square, then there are exactly $\frac{1}{2}d(c)$ ways to choose a, b where $d(x)$ denotes the number of divisors of x. Otherwise, there are $\frac{1}{2}(d(c) + 1)$.

   Hence the answer is $\frac{1}{2}(\sum_{x=1}^{60} d(x) + 7)$. If we consider a 60×60 matrix with entry a_{ij} denoting if $j|i$, then note that our sum is the sum of the rows of this matrix. We can count this instead as the sum of the columns:

   $$\sum_{x=1}^{60} d(x) = \sum_{x=1}^{60} \left\lfloor \frac{60}{x} \right\rfloor$$

   This is much easier to count, and will give

   $$60 + 30 + 20 + 15 + 12 + 10 + 8 + 7 + 6 \cdot 2 + 5 \cdot 2 + 4 \cdot 3 + 3 \cdot 5 + 2 \cdot 10 + 1 \cdot 30 = 261$$

   Hence the total count is $\frac{1}{2}(261 + 7) = 134$.

4. Define a search algorithm called `powSearch`. Throughout, assume A is a 1-indexed sorted array of distinct integers. To search for an integer b in this array, we search the indices $2^0, 2^1, \dots$ until we either reach the end of the array or $A[2^k] > b$. If at any point we get $A[2^k] = b$ we stop and return 2^k. Once we have $A[2^k] > b > A[2^{k-1}]$, we throw away the first 2^{k-1} elements of A, and recursively search in the same fashion. For example, for an integer which is at position 3 we will search the locations $1, 2, 4, 3$.

   Define $g(x)$ to be a function which returns how many (not necessarily distinct) indices we look at when calling `powSearch` with an integer b at position x in A. For example, $g(3) = 4$. If A has length 64, find

   $$g(1) + g(2) + \ldots + g(64).$$

Proposed by Misha Ivkov

Answer: 808

Solution. Note that what `powSearch` does is determines the position of b in binary, MSB first. For all except the last 1 bit, we overshoot by 1 index. For example, to determine the MSB of an element at index 15, we overshoot and examine $1, 2, 4, 8, 16$. Hence if $x = b_k b_{k-1} \dots b_1 b_0$, then

$$g(x) = -1 + \sum_{i=0}^{k} b_i + \sum_{i=0}^{k} (i+1) b_i = -1 + 2\sum_{i=0}^{k} b_i + \sum_{i=0}^{k} i b_i.$$

Indeed, for $x = 3 = 11_2$ this gives $-1 + 4 + 1 = 4$.

Each of $b_0 \dots b_5$ appear in exactly half of the numbers in this range, and b_6 appears in only one. Hence the answer is

$$-1 \cdot 64 + 2 \cdot (6 \cdot 32 + 1) + (0 + 1 + \dots + 5) \cdot 32 + 7 = \boxed{808}.$$

5. In the game of Ric-Rac-Roe, two players take turns coloring squares of a 3×3 grid in their color; a player wins if they complete a row or column of their color on their turn. If Alice and Bob play this game, picking an uncolored square uniformly at random on their turn, what is the probability that they tie?

Proposed by Patrick Lin

Answer: $\frac{5}{14}$

Solution. Without loss of generality, suppose Alice goes first. We count how many of the $\binom{9}{5} = 126$ configurations result in ties; say Alice (or Bob) *beats* a configuration if they have a row or column in their color.

First, observe that if Bob beats a configuration then Alice must also beat it, so it suffices to find the number of configurations that Alice beats with her five squares. This is given by $6 \cdot \binom{6}{2} - 9 = 81$, as we may fix a row or column and then pick two additional squares, accounting for the cases where we have both a row and column. The desired probability is therefore $1 - \frac{81}{126} = \frac{5}{14}$.

6. There are 100 lightbulbs $B_1, \dots, B_{100}$ spaced evenly around a circle in this order. Additionally, there are 100 switches $S_1, \dots, S_{100}$ such that for all $1 \le i \le 100$, switch S_i toggles the states of lights B_{i-1} and B_{i+1} (where here $B_{101} = B_1$). Suppose David chooses whether to flick each switch with probability $\frac{1}{2}$. What is the expected number of lightbulbs which are on at the end of this process given that not all lightbulbs are off?

Proposed by David Altizio

Answer: $\frac{25 \cdot 2^{99}}{2^{98} - 1}$

Note: the original problem erroneously left out the condition that all lightbulbs initially are off. However, the solution below does not depend on this fact, and it is easy to check that the computations remain the same regardless of the initial configuration.

Solution. Let X denote the random variable which equals the number of lightbulbs on at the end of this process, and let A be the event that at least one lightbulb is on. The crucial claim is that

$$\mathbb{E}[X \mid A] = \frac{\mathbb{E}[X]}{\mathbb{P}(A)};$$

indeed, this follows from the fact that $\mathbb{E}[X \mid \bar{A}] = 0$ when combined with the Law of Conditional Expectation.

Computing $\mathbb{E}[X]$ is easy: each lightbulb has probability $\frac{1}{2}$ of being off, so by Linearity of Expectation $\mathbb{E}[X] = 100 \cdot \frac{1}{2} = 50$. It remains to compute the probability that not all bulbs are off. If instead all bulbs are off, then all even-numbered switches must be either on or off, and similarly all odd-numbered switches must be on or off. The position of the first and second switches uniquely determines the rest of the switches, and so $\mathbb{P}(\bar{A}) = 2^{-98}$. Plugging this back in yields a final answer of

$$\frac{25 \cdot 2^{99}}{2^{98} - 1} \approx 50.00000000000000000000000000016.$$

7. Consider the set L of binary strings of length less than or equal to 9, and for a string w define w^+ to be the set $\{w, w^2, w^3, \ldots\}$ where w^k represents w concatenated to itself k times. How many ways are there to pick an ordered pair of (not necessarily distinct) elements $x, y \in L$ such that $x^+ \cap y^+ \neq \varnothing$?

   Proposed by Misha Ivkov

   Answer: 1250

   Solution. We first show a key component of the proof: namely that $x^m = y^n$ for strings x and y only if $x = z^n$ and $y = z^m$ for some string z. First we show actually that $x^m = y^n \Rightarrow xy = yx$. Without loss of generality let $x = yw$. Then

$$(yw)^m = y^n \Rightarrow (wy)^{m-1}w = y^{n-1} \Rightarrow (wy)^m = y^n$$

   so $wy = yw$ and finally $xy = ywy = yx$ as desired.

   Now instead we show $xy = yx$ implies our statement of z^m, z^n. Let's do induction on $|xy|$. If $|xy| = 2$, then $|x| = |y| = 1 \Rightarrow x = y = z$. Else, let $|xy| = k$. Then WLOG $|x| \geq |y|$, so let $x = yw$. Then $wy = yw$ and since $|wy| < |xy|$ we have $w = z^\ell$ and $y = z^m$. Then $x = z^{\ell+m}$ as desired.

   Let's fix $|z| = k$. Then there are $\left\lfloor \frac{9}{k} \right\rfloor^2$ possibilities for the exponents n, m. Now let's compute the number of such z. Let $f(d)$ denote the number of such strings of length d. Note that $2^k = \sum_{d|k} f(d)$, so by Moebius Inversion

$$f(d) = \sum_{d|k} \mu\left(\frac{k}{d}\right) 2^d.$$

   (Manually solving for the values of f using these equations gives the same result.) Finally the answer is

$$\sum_{k=1}^{9} \left\lfloor \frac{9}{k} \right\rfloor^2 \left(\sum_{d|k} \mu\left(\frac{k}{d}\right) 2^d \right) = 1250.$$

   Remark. We can also compute the sum with a similar method to A7.

8. Consider the following graph algorithm (where V is the set of vertices and E the set of edges in G):

```
def s(G):
    if |V| = 0: return true
    for edge (u,v) in E:
        H = G - u - v
        if s(H) = true: return true
    return false
```

   where $G - u - v$ means the subgraph of G which does not contain vertices u, v and all edges using them. How many graphs G with vertex set $\{1, 2, 3, 4, 5, 6\}$ and *exactly* 6 edges satisfy $s(G)$ being true?

   Proposed by Misha Ivkov

   Answer: 2790

   Solution. This algorithm returns true if and only if the graph has a perfect matching. Clearly if the graph does have a perfect matching $\{e_1, e_2, \ldots, e_n\}$ then the algorithm will explore this state space. Now assume the algorithm returns true. We can modify it so that $s(G)$ returns the specific edges removed along the path to no vertices. Then it is easy to see that returning true implies a perfect matching.

   There are $\frac{\binom{6}{2}\binom{4}{2}}{3!} = 15$ possible perfect matchings in a graph with 6 vertices. Then we can use PIE. Note that every pair of perfect matchings intersects in either 0 or 1 edge, and every triple of perfect matchings uses more than 6 edges. There are $\frac{15*6}{2} = 45$ ways to choose two perfect matchings that intersect: this follows because every perfect matching shares an edge with exactly 6 others. In addition, there are $\binom{15}{2} - 45 = 60$ other perfect matching pairs. Then the answer is

$$15\binom{12}{3} - 45\binom{10}{1} - 60\binom{9}{0} = 2790.$$

9. There are 15 cities, and there is a train line between each pair operated by either the Carnegie Rail Corporation or the Mellon Transportation Company. A tourist wants to visit exactly three cities by travelling in a loop, all by travelling on one line. What is the minimum number of such 3-city loops?

   Proposed by Max Aires

   Answer: 88

   Solution. We solve for general n. Let C be the number of loops for the CRC, M be the number for the MTC, and P be the number of loops which have at least one of each rail company. Then $C + M + P = \binom{n}{3}$. Let r_i denote the number of cities that city i is connected to via a CRC line. Then note that we have $r_i(n-1-r_i)$ ways to choose these neighbors and produce a triangle which is not on the same line. Then, $P = \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$ since we overcount the number of triangles by the endpoints. Hence,

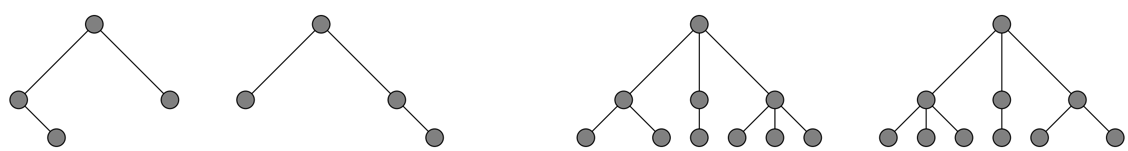
$$C + M = \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$$

   Suppose from here than n is odd for simplicity. Note that we want to minimize $C + M \iff$ maximize P. Note that $r_i(n-1-r_i)$ is maximized where $r_i = \frac{n-1}{2}$, so plugging this in gives that

$$C + M \geq \binom{n}{3} - \frac{n(n-1)^2}{8}$$

   Then plugging in $n = 15$ gives $C+M \geq 87.5$. From here, we construct a solution where $C+M = 88$. Consider the complete bipartite graph on 14 vertices $K_{7,7}$. Denote it as having vertices $a_1, \ldots, a_7, b_1, \ldots, b_7$. Now cut the edges $(a_1, b_1), (a_2, b_2), (a_3, b_3)$, and add a vertex m which is connected to $a_1, b_1, a_2, b_2, a_3, b_3$. This will be our CRC lines, and the remaining edges of K_{15} (complete graph on 15 vertices) are the MTC lines. Then note that $M = 2\binom{7}{3} + 2\binom{4}{2} = 82$, and $C = 3^2 - 3 = 6$ giving a total of 88.

10. Define a *rooted tree* to be a tree T with a singular node designated as the *root* of T. (Note that every node in the tree can have an arbitrary number of children.) Each vertex adjacent to the root node of T is itself the root of some tree called a *maximal subtree* of T. Say two rooted trees T_1 and T_2 are *similar* if there exists some way to cycle the maximal subtrees of T_1 to get T_2. For example, the first pair of trees below are similar but the second pair are not. How many rooted trees with 2019 nodes are there up to similarity?



   Proposed by Gunmay Handa and Misha Ivkov

   Answer: $\frac{\binom{4036}{2018}+\binom{2018}{1009}+8064}{4036}$

   Solution. We first give an explicit bijection between equivalence classes of trees with $n+1$ nodes and equivalence classes necklaces with $2n$ beads having n white and n black under rotation. The idea is that a downstep in the traversal of the tree will produce a white bead on the necklace and an upstep produces a black bead. Then it is clear that cyclic rotation preserves the necklace.

   We use Burnside's to count the number of possible ways to orient this necklace. Note that shifting by k means that $a_0 = a_k = a_{2k} = \ldots = a_{mk}$ for all m. If k is odd, then we will have that $a_0 = a_1 = \ldots a_{2n-1}$. Now consider k with $\gcd(k, 2n) = 2d$. Then we will have $a_0 = a_{2d} = a_{4d} = \ldots = a_{2md}$. Similar equivalence classes exist for each value between 0 and $2d-1$. Then there are $\binom{2d}{d}$ ways to choose d equivalence classes to be black and d to be white. From here, there are $\phi(n/d)$ ways to choose such a k. Hence by Burnside's, the answer is

$$\frac{1}{2n}\sum_{d|n}\phi\left(\frac{n}{d}\right)\binom{2d}{d}$$

wherein we can plug in 2018 for n.