CMIMC 2019

Combinatorics and Computer Science Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty Hall 2302.

Combinatorics and Computer Science

1. Patrick tosses four four-sided dice, each numbered 1 through 4. What's the probability their product is a multiple of four?

2. How many ways are there to color the vertices of a cube red, blue, or green such that no edge connects two vertices of the same color? Rotations and reflections are considered distinct colorings.

3. How many ordered triples (a, b, c) of integers with $1 \leq a \leq b \leq c \leq 60$ satisfy $a \cdot b = c$?

4. Define a search algorithm called `powSearch`. Throughout, assume A is a 1-indexed sorted array of distinct integers. To search for an integer b in this array, we search the indices $2^0, 2^1, \ldots$ until we either reach the end of the array or $A[2^k] > b$. If at any point we get $A[2^k] = b$ we stop and return 2^k. Once we have $A[2^k] > b > A[2^{k-1}]$, we throw away the first 2^{k-1} elements of A, and recursively search in the same fashion. For example, for an integer which is at position 3 we will search the locations $1, 2, 4, 3$.

   Define $g(x)$ to be a function which returns how many (not necessarily distinct) indices we look at when calling `powSearch` with an integer b at position x in A. For example, $g(3) = 4$. If A has length 64, find

   $$g(1) + g(2) + \ldots + g(64).$$

5. In the game of Ric-Rac-Roe, two players take turns coloring squares of a 3×3 grid in their color; a player wins if they complete a row or column of their color on their turn. If Alice and Bob play this game, picking an uncolored square uniformly at random on their turn, what is the probability that they tie?

6. There are 100 lightbulbs $B_1, \ldots, B_{100}$ spaced evenly around a circle in this order. Additionally, there are 100 switches $S_1, \ldots, S_{100}$ such that for all $1 \leq i \leq 100$, switch S_i toggles the states of lights B_{i-1} and B_{i+1} (where here $B_{101} = B_1$). Suppose David chooses whether to flick each switch with probability $\frac{1}{2}$. What is the expected number of lightbulbs which are on at the end of this process given that not all lightbulbs are off?

7. Consider the set L of binary strings of length less than or equal to 9, and for a string w define w^+ to be the set $\{w, w^2, w^3, \ldots\}$ where w^k represents w concatenated to itself k times. How many ways are there to pick an ordered pair of (not necessarily distinct) elements $x, y \in L$ such that $x^+ \cap y^+ \neq \varnothing$?

8. Consider the following graph algorithm (where V is the set of vertices and E the set of edges in G).

```
procedure S(G)
    if |V| = 0 then return true
    for (u, v) in E do
        H ← G − u − v
        if S(H) then return true
    return false
```

   Here $G - u - v$ means the subgraph of G which does not contain vertices u, v and all edges using them. How many graphs G with vertex set $\{1, 2, 3, 4, 5, 6\}$ and *exactly* 6 edges satisfy $s(G)$ being true?

9. There are 15 cities, and there is a train line between each pair operated by either the Carnegie Rail Corporation or the Mellon Transportation Company. A tourist wants to visit exactly three cities by travelling in a loop, all by travelling on one line. What is the minimum number of such 3-city loops?

10. Define a *rooted tree* to be a tree T with a singular node designated as the *root* of T. (Note that every node in the tree can have an arbitrary number of children.) Each vertex adjacent to the root node of T is itself the root of some tree called a *maximal subtree* of T. Say two rooted trees T_1 and T_2 are *similar* if there exists some way to cycle the maximal subtrees of T_1 to get T_2. For example, the first pair of trees below are similar but the second pair are not. How many rooted trees with 2019 nodes are there up to similarity?