CMIMC 2019

Algebra and Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty Hall 2302 by the end of lunch.

Algebra and Number Theory

1. Let a_1, a_2, …, a_n be a geometric progression with $a_1 = \sqrt{2}$ and $a_2 = \sqrt[3]{3}$. What is

$$\frac{a_1 + a_{2013}}{a_7 + a_{2019}}?$$

2. For all positive integers n, let $f(n)$ return the smallest positive integer k for which $\frac{n}{k}$ is not an integer. For example, $f(6) = 4$ because 1, 2, and 3 all divide 6 but 4 does not. Determine the largest possible value of $f(n)$ as n ranges over the set $\{1, 2, \ldots, 3000\}$.

3. Let $P(x)$ be a quadratic polynomial with real coefficients such that $P(3) = 7$ and

$$P(x) = P(0) + P(1)x + P(2)x^2$$

for all real x. What is $P(-1)$?

4. Determine the sum of all positive integers n between 1 and 100 inclusive such that

$$\gcd(n, 2^n - 1) = 3.$$

5. Let x_n be the smallest positive integer such that 7^n divides $x_n^2 - 2$. Find $x_1 + x_2 + x_3$.

6. Let a, b and c be the distinct solutions to the equation $x^3 - 2x^2 + 3x - 4 = 0$. Find the value of

$$\frac{1}{a(b^2+c^2-a^2)} + \frac{1}{b(c^2+a^2-b^2)} + \frac{1}{c(a^2+b^2-c^2)}.$$

7. For all positive integers n, let

$$f(n) = \sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2.$$

Compute $f(2019) - f(2018)$. Here $\varphi(n)$ denotes the number of positive integers less than or equal to n which are relatively prime to n.

8. It is given that the roots of the polynomial $P(z) = z^{2019} - 1$ can be written in the form $z_k = x_k + iy_k$ for $1 \leq k \leq 2019$. Let Q denote the monic polynomial with roots equal to $2x_k + iy_k$ for $1 \leq k \leq 2019$. Compute $Q(-2)$.

9. Let $a_0 = 29$, $b_0 = 1$ and

$$a_{n+1} = a_n + a_{n-1} \cdot b_n^{2019}, \qquad b_{n+1} = b_n b_{n-1}$$

for $n \geq 1$. Determine the smallest positive integer k for which 29 divides $\gcd(a_k, b_k - 1)$ whenever a_1, b_1 are positive integers and 29 does not divide b_1.

10. Let $\varphi(n)$ denotes the number of positive integers less than or equal to n which are relatively prime to n. Determine the number of positive integers $2 \leq n \leq 50$ such that all coefficients of the polynomial

$$\left(x^{\varphi(n)} - 1\right) - \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (x - k)$$

are divisible by n.