CMIMC 2018

Team Round

Set One

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 problems to be solved in 30 minutes. The problems are not ordered.
3. Write your team name and team ID on both answer sheets. Choose one to be the ten-minute submission and the other to be the thirty-minute submission, and circle the appropriate choice.
4. Each problem is worth up to 8 points. Points earned per problem are halved for a thirty-minute submission and quartered if only one leg is correct.
5. Write your answers in the corresponding boxes on the answer sheets.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to the registration desk on the first floor of the University Center by 1:30 PM.

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CMIMC 2018

Team (Set One)

1-1. Let ABC be a triangle with $BC = 30$, $AC = 50$, and $AB = 60$. Circle ω_B is the circle passing through A and B tangent to BC at B; ω_C is defined similarly. Suppose the tangent to $\odot(ABC)$ at A intersects ω_B and ω_C for the second time at X and Y respectively. Compute XY.

2-1. Suppose that a and b are non-negative integers satisfying $a + b + ab + a^b = 42$. Find the sum of all possible values of $a + b$.

3-1. Let Ω be a semicircle with endpoints A and B and diameter 3. Points X and Y are located on the boundary of Ω such that the distance from X to AB is $\frac{5}{4}$ and the distance from Y to AB is $\frac{1}{4}$. Compute

$$(AX + BX)^2 - (AY + BY)^2.$$

4-1. Define an integer $n \geq 0$ to be *two-far* if there exist integers a and b such that a, b, and $n + a + b$ are all powers of two. If N is the number of two-far integers less than 2048, find the remainder when N is divided by 100.

5-1. How many ordered triples (a, b, c) of integers satisfy the inequality

$$a^2 + b^2 + c^2 \leq a + b + c + 2?$$

6-2. Let $T = TNYWR$. Compute the number of ordered triples (a, b, c) such that a, b, and c are distinct positive integers and $a + b + c = T$.

7-2. Let $T = TNYWR$. A total of $2T$ students go on a road trip. They take two cars, each of which seats T people. Call two students *friendly* if they sat together in the same car going to the trip and in the same car going back home. What is the smallest possible number of friendly pairs of students on the trip?

8-2. Let $T = TNYWR$, and let $T = 10X + Y$ for an integer X and a digit Y. Suppose that a and b are real numbers satisfying $a + \frac{1}{b} = Y$ and $\frac{b}{a} = X$. Compute $(ab)^4 + \frac{1}{(ab)^4}$.

9-2. Let $T = TNYWR$. The solutions in z to the equation

$$\left(z + \frac{T}{z}\right)^2 = 1$$

form the vertices of a quadrilateral in the complex plane. Compute the area of this quadrilateral.

10-2. Let $T = TNYWR$. Circles ω_1 and ω_2 intersect at P and Q. The common external tangent ℓ to the two circles closer to Q touches ω_1 and ω_2 at A and B respectively. Line AQ intersects ω_2 at X while BQ intersects ω_1 again at Y. Let M and N denote the midpoints of $\overline{AY}$ and $\overline{BX}$, also respectively. If $AQ = \sqrt{T}$, $BQ = 7$, and $AB = 8$, then find the length of MN.

CMIMC 2018

Team Round

Set Two

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Team (Set Two)

1-2. Let $T = TNYWR$. For some positive integer k, a circle is drawn tangent to the coordinate axes such that the lines $x+y=k^2, x+y=(k+1)^2, \dots, x+y=(k+T)^2$ all pass through it. What is the minimum possible value of k?

2-2. Let $T = TNYWR$. Suppose that a sequence $\{a_n\}$ is defined via $a_1 = 11, a_2 = T$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Find $a_{19} + a_{20}$.

3-2. Let $T = TNYWR$. T people each put a distinct marble into a bag; its contents are mixed randomly and one marble is distributed back to each person. Given that at least one person got their own marble back, what is the probability that everyone else also received their own marble?

4-2. Let $T = TNYWR$. Let CMU be a triangle with $CM = 13$, $MU = 14$, and $UC = 15$. Rectangle $WEAN$ is inscribed in $\triangle CMU$ with points W and E on $\overline{MU}$, point A on $\overline{CU}$, and point N on $\overline{CM}$. If the area of $WEAN$ is T, what is its perimeter?

5-2. Let $T = TNYWR$. David rolls a standard T-sided die repeatedly until he first rolls T, writing his rolls in order on a chalkboard. What is the probability that he is able to erase some of the numbers he's written such that all that's left on the board are the numbers $1, 2, \dots, T$ in order?

6-1. Jan rolls a fair six-sided die and calls the result r. Then, he picks real numbers a and b between 0 and 1 uniformly at random and independently. If the probability that the polynomial $\frac{x^2}{r} - x\sqrt{a} + b$ has a real root can be expressed as simplified fraction $\frac{p}{q}$, find p.

7-1. Let $ABCD$ be a unit square, and suppose that E and F are on $\overline{AD}$ and $\overline{AB}$ such that $AE = AF = \frac{2}{3}$. Let $\overline{CE}$ and $\overline{DF}$ intersect at G. If the area of $\triangle CFG$ can be expressed as simplified fraction $\frac{p}{q}$, find $p+q$.

8-1. Let $\triangle ABC$ be a triangle with $AB = 3$ and $AC = 5$. Select points $D, E,$ and F on $\overline{BC}$ in that order such that $\overline{AD} \perp \overline{BC}$, $\angle BAE = \angle CAE$, and $\overline{BF} = \overline{CF}$. If E is the midpoint of segment $\overline{DF}$, what is BC^2?

9-1. Andy rolls a fair 4-sided dice, numbered 1 to 4, until he rolls a number that is less than his last roll. If the expected number of times that Andy will roll the dice can be expressed as a reduced fraction $\frac{p}{q}$, find $p+q$.

10-1. Find the smallest positive integer k such that $\underbrace{11\dots11}_{k\text{ 1's}}$ is divisible by 9999.