Number Theory Solutions Packet

1. Suppose a, b, and c are relatively prime integers such that

$$\frac{a}{b+c} = 2 \qquad \text{and} \qquad \frac{b}{a+c} = 3.$$

What is the value of $|c|$?

Proposed by David Altizio

Solution. The given equations rewrite to $2c = a - 2b$ and $3c = b - 3a$, which implies

$$3(a-2b) = 2(b-3a) \quad \Rightarrow \quad 9a = 8b.$$

Hence $a = \pm 8$ and $b = \pm 9$. Now back-substitution yields $c = \pm 5$, giving an answer of $\boxed{5}$.

2. Find all integers n for which $(n-1)\cdot 2^n + 1$ is a perfect square.

Proposed by Cody Johnson

Solution. First note that if $n \le 0$, then $(n-1)2^n$ is an integer precisely when $n \ge -1$; checking yields $n = 0$ and $n = -1$ as solutions. Now assume $n > 0$. We need to solve

$$x^2 = (n-1)\cdot 2^n + 1$$

or

$$(x-1)(x+1) = (n-1)\cdot 2^n.$$

Note that $\gcd(x-1, x+1) \le 2$, so 2^{n-1} completely divides one either $x-1$ or $x+1$. Supposing that $2^{n-1} \mid x-1$, we have

$$2 = (x+1) - (x-1) \le 2(n-1) - 2^{n-1},$$

since $x - 1 \ge 2^{n-1}$, and so $x + 1 \le 2(n-1)$. For $n > 4$, this is impossible because $2(n-1) - 2^{n-1} < 0$. On the other hand, if $2^{n-1} \mid x+1$, then we have

$$2 = (x+1) - (x-1) \ge 2^{n-1} - 2(n-1) > 2$$

for all $n > 4$. In either case, $n > 4$ is impossible, so we only need to test $n \le 4$. We get $n = \boxed{-1, 0, 1, 4}$ are the answers.

3. Let S be the set of natural numbers that cannot be written as the sum of three squares. Legendre's three-square theorem states that $S = \{4^a \cdot (8b+7) \mid a, b \ge 0\}$. Find the smallest $n \in \mathbb{N}$ such that n and $n+1$ are both in S.

Proposed by Cody Johnson

Solution. If n is even, then $4 \mid n$, so $n + 1 \equiv 1 \pmod 4$ which is not $7 \pmod 8$, so it is not in S. Thus, n is odd, so $8 \mid n+1$, so $16 \mid n+1$, so $n + 1 \ge 16 \cdot (8 \cdot 0 + 7) = 112$. Thus, $n \ge \boxed{111}$, which we can easily verify works since $n = 4^0 \cdot (8 \cdot 13 + 7) \in S$ and $n + 1 = 4^2 \cdot (8 \cdot 0 + 7) \in S$.

4. Let $a > 1$ be a positive integer. The sequence of natural numbers $\{a_n\}$ is defined as follows: $a_1 = a$ and for all $n \ge 1$, a_{n+1} is the largest prime factor of $a_n^2 - 1$. Determine the smallest possible value of a such that the numbers $a_1, a_2, \ldots, a_7$ are all distinct.

Proposed by David Altizio

Solution. First remark that if $a = 2$, then the sequence repeats $2 \mapsto 3 \mapsto 2 \mapsto \cdots$, so in order to minimize a_7 it must be the case that $a_7 = 2$ and $a_6 \ge 3$. (Note that the other way around is not possible, since for no integer $a \ge 4$ is $a^2 - 1$ a power of 2.) Now examine a_2, noting that it is prime. Then a_3 must satisfy

$$a_3 \mid a_2^2 - 1 = (a_2 - 1)(a_2 + 1).$$

Since a_2 is an odd prime, $a_2 - 1$ and $a_2 + 1$ are both even, and so $a_3 \le \frac{a_2+1}{2}$. Thus

$$a_2 \ge 2a_3 - 1 \ge 4a_4 - 3 \ge \cdots \ge 16a_6 - 15 \ge 33,$$

where here we use the fact that $a_6 \ge 3$. Trying a few primes past 33 shows that in fact

$$47 \mapsto 23 \mapsto 11 \mapsto 5 \mapsto 3 \mapsto 2$$

gives a valid sequence $a_2, \ldots, a_7$ of distinct integers. Hence the smallest possible value of a_2 is 47, meaning the smallest possible value of a_1 is $\boxed{46}$.

5. It is given that there exist unique integers $m_1, \ldots, m_{100}$ such that

$$0 \le m_1 < m_2 < \cdots < m_{100} \quad \text{and} \quad 2018 = \binom{m_1}{1} + \binom{m_2}{2} + \cdots + \binom{m_{100}}{100}.$$

Find $m_1 + m_2 + \cdots + m_{100}$.

Proposed by David Altizio

Solution. Say the sequence *jumps* at i if $m_{i+1} - m_i > 1$. If $m_{100} \ge 102$, then $\binom{m_{100}}{100} \ge \binom{102}{100} = 5151 > 2018$. Thus, the sequence jumps at most twice, i.e., for some $1 \le a \le b \le 100$, we have $m_i = i - 1$ for all $1 \le i \le a$, $m_i = i$ for all $a < i \le b$, and $m_i = i + 1$ for all $b < i \le 100$. Hence, we have

$$2018 = \sum_{i=1}^{a} \binom{i-1}{i} + \sum_{i=a+1}^{b} \binom{i}{i} + \sum_{i=b+1}^{100} \binom{i+1}{i} = b - a + \frac{101(102)}{2} - \frac{(b+1)(b+2)}{2},$$

so

$$3132 = \frac{b^2 + b}{2} + a.$$

Trying some values of b near $\sqrt{2 \cdot 3132} \approx \sqrt{6400} = 80$, we find that $b = 78$, $a = 51$ works. Thus, the answer is

$$\sum_{i=1}^{51} (i-1) + \sum_{i=52}^{78} i + \sum_{i=79}^{100} (i+1) = \frac{100(101)}{2} - 51 + 22 = \boxed{5021}.$$

Remark. This is called the 100-nomial representation of 2018. In general, for any positive integers m and n, one can show that the m-nomial representation of n is unique.

6. Let $\phi(n)$ denote the number of positive integers less than or equal to n that are coprime to n. Find the sum of all $1 < n < 100$ such that $\phi(n) | n$.

Proposed by Andrew Kwon

Solution. We claim that for $n > 1$, $\phi(n) | n \iff n = 2^a 3^b$, where $a \ge 1$ and $b \ge 0$. Evidently n must be even. Let $n = 2^a m$, where m is odd. If m has more than 2 prime distinct prime factors, then $\varphi(m)$ will be divisible by 4. However, then $2^{a+1} | 2^{a-1} \varphi(m) = \varphi(n) | n$, which is a contradiction. Therefore, $m = p^b$ for some prime p and nonnegative integer b. Then, $p - 1 | \varphi(n) | n$, and so $p - 1$ must be a power of 2. Upon analogous considerations as before to the largest power of 2 that can divide $\varphi(n)$, we find that $p - 1$ is necessarily equal to 2, and so $p = 3$.

We thus must find the sum of all integers of the form $2^a 3^b < 100$, where $a \ge 1$ and $b \ge 0$, and casing on the value of b we can calculate this with geometric series to be $\boxed{492}$.

7. For each $q \in \mathbb{Q}$, let $\pi(q)$ denote the period of the repeating base-16 expansion of q, with the convention of $\pi(q) = 0$ if q has a terminating base-16 expansion. Find the maximum value among

$$\pi\left(\frac{1}{1}\right), \ \pi\left(\frac{1}{2}\right), \ \ldots, \ \pi\left(\frac{1}{70}\right).$$

Proposed by Cody Johnson

Solution. Suppose $\frac{1}{n}$ has a repeating base-16 expansion with period π. If we multiply $\frac{1}{n}$ by a large enough power of 16 (say 16^N), then the fractional part will look like $0.\overline{b_1 \dots b_\pi}$. If we then multiply this by just 16^π and take the difference, we will get an integer, i.e., $16^{N+\pi}\frac{1}{n} - 16^N\frac{1}{n} = \frac{16^{N+\pi}-16^N}{n} \in \mathbb{Z}$. This proves that the length of the period of $\frac{1}{n}$ is equal to the smallest integer p such that $n \mid 16^{N+\pi} - 16^N$ for some sufficiently large N, or equivalently the smallest π such that

$$16^{N+\pi} \equiv 16^N \pmod n \implies 16^\pi \equiv 1 \pmod n$$

(since $\gcd(16, n) = 1$).

When n is odd, π is equal to the multiplicative order of $16 \pmod n$. However, $16 = 2^4$, so we need $2^{4k} \equiv 1 \pmod n$ for the smallest k possible. Note that $2^{2\phi(n)} \equiv 1 \pmod n$ and $4 \mid 2\phi(n)$ since $\phi(n)$ is even. Thus,

$$\pi \le \frac{2\phi(n)}{4} \le \frac{n-1}{2} \implies \pi \le \left\lfloor \frac{n-1}{2} \right\rfloor \le \left\lfloor \frac{68-1}{2} \right\rfloor = \boxed{33}$$

as long as $n \le 68$. When $n = 69$, note that $16^{11} \equiv 1 \pmod{69}$. When $n = 67$, which is prime, we can get prove that we have equality for this inequality by showing that 2 is a primitive root $\pmod{67}$. It suffices to show that $2^{33}, 2^{22}, 2^6 \not\equiv 1 \pmod{67}$, which is fairly simple.

8. It is given that there exists a unique triple of positive primes (p, q, r) such that $p < q < r$ and

$$\frac{p^3 + q^3 + r^3}{p + q + r} = 249.$$

Find r.

Proposed by David Altizio

Solution. We recall the identity $p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp)$. Hence,

$$(p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) = p^3 + q^3 + r^3 - 3pqr = 249(p + q + r) - 3pqr$$

$$\implies 3pqr = (p + q + r)(249 + pq + qr + pr - p^2 - q^2 - r^2)$$

The left hand side is a product of primes, so there are only a finite number of ways we can assign these primes to the factors on right hand side. Note that $p + q + r > 3$ and $p + q + r > 3p$, so the first thing we try is setting $p + q + r = 3q$. Then

$$0 = 249 + q(p + r) - p^2 - q^2 - r^2 = 249 + q^2 - p^2 - r^2$$

which implies $3p^2 - 2pr + 3r^2 = 996$. Consequently, $3|2pr$ and since $r > p$, we get that $p = 3$; plugging this into the newly derived equation gives $r = \boxed{19}$. It is not hard to verify that $(p, q, r) = (3, 11, 19)$ is indeed a valid triple.

9. Let $\phi(n)$ denote the number of positive integers less than or equal to n which are coprime to n. Find the value of

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{5^n + 1}.$$

Proposed by Gunmay Handa

Solution. Let $x = \frac{1}{5}$. Then

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{x^{-n} + 1} = \sum_{n=1}^{\infty} \frac{\phi(n)}{x^{-n} - 1} - 2\sum_{n=1}^{\infty} \frac{\phi(n)}{x^{-2n} - 1} = \sum_{n=1}^{\infty} \frac{\phi(n)x^n}{1 - x^n} - 2\sum_{n=1}^{\infty} \frac{\phi(n)x^{2n}}{1 - x^{2n}}.$$

The key claim is then that $\sum_{n=1}^{\infty} \frac{\phi(n)t^n}{1-t^n} = \frac{t}{(1-t)^2}$ for $|t| < 1$. We have

$$\sum_{n=1}^{\infty} \frac{\phi(n)t^n}{1-t^n} = \sum_{n=1}^{\infty} \phi(n) \sum_{m=1}^{\infty} t^{nm} = \sum_{s=1}^{\infty} st^s = \frac{t}{(1-t)^2}$$

where we used the fact that $\sum_{d|n} \phi(d) = n$. Finally, the desired value is just

$$\frac{x}{(1-x)^2} - 2\frac{x^2}{(1-x^2)^2} = \frac{x(1+x^2)}{(1-x^2)^2} = \boxed{\frac{65}{288}}.$$

10. Let $a_1 < a_2 < \cdots < a_k$ denote the sequence of all positive integers between 1 and 91 which are relatively prime to 91, and set $\omega = e^{2\pi i/91}$. Define

$$S = \prod_{1 \le q < p \le k} \left(\omega^{a_p} - \omega^{a_q}\right).$$

Given that S is a positive integer, compute the number of positive divisors of S.

Proposed by David Altizio

Solution. Let $\Phi_n(x)$ be the n^{th} cyclotomic polynomial. Let S be the desired product and for each $1 \le i \le k$ define $P_i(x) = \frac{\Phi_{91}(x)}{x-\omega^{a_i}}$. Then we have

$$S^2 = \prod_{p \ne q} (\omega^{a_q} - \omega^{a_p}) = \prod_{i=1}^{k} P_i(\omega^{a_i}).$$

Since $\Phi_{91}(\omega^{a_i}) = 0$ by definition, L'Hopital's rule gives $P_i(\omega^{a_i}) = \Phi_{91}'(\omega^{a_i})$. Now by well-known properties of cyclotomic polynomials,

$$\Phi_{91}(x) = \frac{x^{91}-1}{\Phi_1(x)\Phi_7(x)\Phi_{13}(x)} = \frac{(x^{91}-1)(x-1)}{(x^7-1)(x^{13}-1)}.$$

Since $(\omega^{a_i})^{91} = 1$ for all i, we have by the product rule that

$$\Phi_{91}'(\omega^{a_i}) = \frac{d}{dx}\left[(x^{91}-1) \cdot \frac{(x-1)}{(x^7-1)(x^{13}-1)}\right]_{x=\omega^{a_i}} = 91(\omega^{a_i})^{90} \cdot \frac{\omega^{a_i}-1}{((\omega^{a_i})^7-1)((\omega^{a_i})^{13}-1)}.$$

We of course have that $\prod_i (1-\omega^{a_i}) = \Phi_{91}(1)$. Note that the sequence $((\omega^{a_i})^7)_{1 \le i \le k}$ must contain each of the twelve nontrivial 13^{th} roots of unity exactly six times. Hence $\prod_i (1-(\omega^{a_i})^7) = \Phi_{13}(1)^6$. Similarly, $\prod_i (1-(\omega^{a_i})^{13}) = \Phi_7(1)^{12}$. Since $\prod_i \omega^{a_i} = 1$ (each root of unity has a conjugate pair, and $\gcd(a_i, 91) = 1 \Leftrightarrow \gcd(91-a_i, 91) = 1$), it follows that

$$|S^2| = \frac{91^{\varphi(91)}\Phi_{91}(1)}{\Phi_7(1)^{12}\Phi_{13}(1)^6}.$$

We have $\Phi_7(1) = 7$, $\Phi_{13}(1) = 13$, $\varphi(91) = 6 \cdot 12 = 72$, and

$$\Phi_{91}(1) = \lim_{x\to 1} \frac{x^{91}-1}{x^{13}-1} \cdot \frac{x-1}{x^7-1} = 7 \cdot \frac{1}{7} = 1.$$

So $|S|^2 = 7^{72-12}13^{72-6}$ and $|S| = 7^{30}13^{33}$, giving a final answer of $31 \cdot 34 = \boxed{1054}$.

Remark. It is possible to do the computations above without using calculus. For example, another solution which is longer but more elementary is to employ PIE + complementary counting, since the above product excludes all terms of the form $\omega^{i_0} - \omega^{j_0}$ where $i_0 j_0$ is a multiple of 7 or 13. (This was the author's original solution.)