CMIMC 2018
Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to the registration desk on the first floor of the University Center by the end of lunch.

Number Theory

1. Suppose a, b, and c are relatively prime integers such that

$$\frac{a}{b+c} = 2 \qquad \text{and} \qquad \frac{b}{a+c} = 3.$$

What is $|c|$?

2. Find all integers n for which $(n-1) \cdot 2^n + 1$ is a perfect square.

3. Let S be the set of natural numbers that cannot be written as the sum of three squares. Legendre's three-square theorem states that S consists of precisely the integers of the form $4^a(8b+7)$ where a and b are nonnegative integers. Find the smallest $n \in \mathbb{N}$ such that n and $n+1$ are both in S.

4. Let $a > 1$ be a positive integer. The sequence of natural numbers $\{a_n\}_{n\geq 1}$ is defined such that $a_1 = a$ and for all $n \geq 1$, a_{n+1} is the largest prime factor of $a_n^2 - 1$. Determine the smallest possible value of a such that the numbers a_1, a_2,..., a_7 are all distinct.

5. It is given that there exist unique integers $m_1, \ldots, m_{100}$ such that

$$0 \leq m_1 < m_2 < \cdots < m_{100} \quad \text{and} \quad 2018 = \binom{m_1}{1} + \binom{m_2}{2} + \cdots + \binom{m_{100}}{100}.$$

Find $m_1 + m_2 + \cdots + m_{100}$.

6. Let $\phi(n)$ denote the number of positive integers less than or equal to n that are coprime to n. Find the sum of all $1 < n < 100$ such that $\phi(n) \mid n$.

7. For each $q \in \mathbb{Q}$, let $\pi(q)$ denote the period of the repeating base-16 expansion of q, with the convention of $\pi(q) = 0$ if q has a terminating base-16 expansion. Find the maximum value among

$$\pi\left(\frac{1}{1}\right), \ \pi\left(\frac{1}{2}\right), \ \ldots, \ \pi\left(\frac{1}{70}\right).$$

8. It is given that there exists a unique triple of positive primes (p, q, r) such that $p < q < r$ and

$$\frac{p^3 + q^3 + r^3}{p + q + r} = 249.$$

Find r.

9. Let $\phi(n)$ denote the number of positive integers less than or equal to n that are coprime to n. Compute

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{5^n + 1}.$$

10. Let $a_1 < a_2 < \cdots < a_k$ denote the sequence of all positive integers between 1 and 91 which are relatively prime to 91, and set $\omega = e^{2\pi i/91}$. Define

$$S = \prod_{1 \leq q < p \leq k} \left(\omega^{a_p} - \omega^{a_q}\right).$$

Given that S is a positive integer, compute the number of positive divisors of S.