CMIMC 2018

Geometry Solutions Packet

1. Let ABC be a triangle. Point P lies in the interior of ABC such that $\angle ABP = 20^\circ$ and $\angle ACP = 15^\circ$. Compute $\angle BPC - \angle BAC$.

   Proposed by David Altizio

   Solution. Note that
   $$\angle BPC + \angle PBC + \angle PCB = 180^\circ = \angle BAC + (20^\circ + \angle PBC) + (15^\circ + \angle PCB);$$
   canceling $\angle PBC + \angle PCB$ from both sides and rearranging yields the desired answer of $\boxed{35^\circ}$.

2. Let $ABCD$ be a square of side length 1, and let P be a variable point on $\overline{CD}$. Denote by Q the intersection point of the angle bisector of $\angle APB$ with $\overline{AB}$. The set of possible locations for Q as P varies along $\overline{CD}$ is a line segment; what is the length of this segment?

   Proposed by David Altizio

   Solution. Note that
   $$\frac{PB}{PA} = \sqrt{\frac{PC^2 + CB^2}{PD^2 + DA^2}} = \sqrt{\frac{PC^2 + 1}{(1 - PC)^2 + 1}}.$$
   This increases as PC increases, so by the Angle Bisector Theorem $\frac{QB}{QA}$ increases as well. It follows that the endpoints of this line segment occur precisely when $P = C$ or $P = B$.

   Let Q_0 be the foot of the angle bisector of $\angle ACD$. By another use of the Angle Bisector Theorem,
   $$\sqrt{2} = \frac{AQ_0}{Q_0B} = \frac{1 - Q_0B}{Q_0B} \quad \Rightarrow \quad Q_0B = \sqrt{2} - 1.$$
   Similarly, if Q_1 is the foot of the angle bisector from $\angle ADB$, $AQ_1 = \sqrt{2} - 1$. It follows that the length of the desired line segment is
   $$1 - 2(\sqrt{2} - 1) = \boxed{3 - 2\sqrt{2}}.$$

3. Let ABC be a triangle with side lengths 5, $4\sqrt{2}$, and 7. What is the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$?

   Proposed by David Altizio

   Solution. Let R be the circumradius of $\triangle ABC$. The key is to realize that by the Extended Law of Sines,
   $$\sin A = \frac{a}{2R}, \quad \sin B = \frac{b}{2R}, \quad \text{and} \quad \sin C = \frac{c}{2R}.$$
   It follows that the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$ is similar to $\triangle ABC$ with scale factor $\frac{1}{2R}$. Thus, it suffices to compute the area of $\triangle ABC$ and divide by $4R^2$ to get the answer.

   WLOG let $AB = 4\sqrt{2}$, $AC = 5$, and $BC = 7$. Let D denote the foot of the altitude from A to BC. The Law of Cosines applied to $\triangle ABC$ yields
   $$\cos C = \frac{5^2 + 7^2 - (4\sqrt{2})^2}{2 \cdot 5 \cdot 7} = \frac{3}{5},$$
   so $\sin C = \frac{4}{5}$ and $AD = 4$. This means that the area of $\triangle ABC$ is $\frac{1}{2} \cdot 4 \cdot 7 = 14$ and
   $$R = \frac{4\sqrt{2}}{2\sin C} = \frac{4\sqrt{2}}{8/5} = \frac{5}{\sqrt{2}}.$$
   Thus the desired answer is
   $$\frac{14}{4 \cdot (\frac{5}{\sqrt{2}})^2} = \boxed{\frac{7}{25}}.$$

4. Suppose $\overline{AB}$ is a segment of unit length in the plane. Let $f(X)$ and $g(X)$ be functions of the plane such that f corresponds to rotation about A 60° counterclockwise and g corresponds to rotation about B 90° clockwise. Let P be a point with $g(f(P)) = P$; what is the sum of all possible distances from P to line AB?

   Proposed by Gunmay Handa

   Solution. In this solution, all angles are directed. For any line ℓ, let $\ell_1 = f(\ell), \ell_2 = g(\ell_1)$. Then $\angle(\ell, \ell_1) = 60^\circ$ and $\angle(\ell_1, \ell_2) = -90^\circ$, so $\angle(\ell, \ell_2) = -30^\circ$. This tells us that the composition of these two rotations itself corresponds to a rotation of 30° clockwise; evidently, rotations can only have one fixed point, which is their center. Construct the point C with $\angle(CA, AB) = 30^\circ$ and $\angle(CB, BA) = -135^\circ$; it is not hard to see that $g(f(C)) \equiv C$ (in this instance, each function is equivalent to reflection about AB). Let C' be the projection of C onto AB, then $CC' \cot 30^\circ - CC' \cot 45^\circ = AC' - BC' = 1$ and so

$$CC' = \frac{1}{\cot 30^\circ - \cot 45^\circ} = \frac{1}{\sqrt{3}-1} = \boxed{\frac{1+\sqrt{3}}{2}}.$$

5. Select points T_1, T_2 and T_3 in $\mathbb{R}^3$ such that $T_1 = (0,1,0)$, T_2 is at the origin, and $T_3 = (1,0,0)$. Let T_0 be a point on the line $x = y = 0$ with $T_0 \neq T_2$. Suppose there exists a point X in the plane of $\triangle T_1T_2T_3$ such that the quantity $(XT_i)[T_{i+1}T_{i+2}T_{i+3}]$ is constant for all $i = 0$ to $i = 3$, where $[\mathcal{P}]$ denotes area of the polygon $\mathcal{P}$ and indices are taken modulo 4. What is the magnitude of the z-coordinate of T_0?

   Proposed by Gunmay Handa

   Solution. Let M be the midpoint of $\overline{T_1T_3}$. We claim that X is the reflection M' of M across T_2. To prove this, first remark that

$$\frac{XT_1}{XT_3} = \frac{[T_1T_2T_0]}{[T_2T_3T_0]} = \frac{\frac{1}{2}(T_1T_2)(T_2T_0)}{\frac{1}{2}(T_2T_3)(T_2T_0)} = \frac{T_1T_2}{T_3T_2} = 1.$$

   Thus $XT_1 = XT_3$; since X lies in the plane $T_1T_2T_3$, X must lie on the perpendicular bisector of $\overline{T_1T_3}$. Now applying similar logic on the points T_2 and T_0 yields

$$\frac{XT_2}{XT_0} = \frac{[T_1T_2T_3]}{[T_1T_3T_0]} = \frac{T_2M}{T_0M}.$$

   Now remark that as X moves farther away from T_2, the ratio $\frac{XT_2}{XT_0}$ gets closer to 1 (without ever equalling one). Thus, either $X \equiv M$ or $X \equiv M'$. To show it is not M, suppose it were, and write

$$\frac{MT_1}{MT_0} = \frac{[T_1T_2T_3]}{[T_2T_3T_0]} = \frac{T_2T_1}{T_2T_0}. \qquad (*)$$

   But this is not possible, since $MT_1 < T_2T_1$ and $MT_0 > T_2T_0$. Thus the only possible option is $T \equiv M'$ as desired.

   Now let z be the z-coordinate of T_0. Using $(*)$ but with M' in place of M gives

$$\frac{1}{z} = \frac{T_2T_1}{T_2T_0} = \frac{M'T_1}{M'T_0} = \frac{\sqrt{5/2}}{\sqrt{1/2+z^2}} = \sqrt{\frac{5}{1+2z^2}};$$

   solving this equation yields $z = \boxed{\frac{\sqrt{3}}{3}}$.

6. Let ω_1 and ω_2 be intersecting circles in the plane with radii 12 and 15, respectively. Suppose Γ is a circle such that ω_1 and ω_2 are internally tangent to Γ at X_1 and X_2, respectively. Similarly, ℓ is a line that is tangent to ω_1 and ω_2 at Y_1 and Y_2, respectively. If $X_1X_2 = 18$ and $Y_1Y_2 = 9$, what is the radius of Γ?

   Proposed by Gunmay Handa

   Solution. First we compute the distance between their centers, which is just

$$\sqrt{9^2 + (15-12)^2} = 3\sqrt{10}.$$

Now let O_1, O_2, O be the centers of ω_1, ω_2, and Γ, respectively, and let $R := OX_1 = OX_2$. Since $\angle X_1OX_2 = \angle O_1OO_2$, we have

$$\frac{R^2+R^2-18^2}{R^2} = \frac{OX_1^2+OX_2^2-X_1X_2^2}{OX_1\cdot OX_2} = \frac{OO_1^2+OO_2^2-O_1O_2^2}{OO_1\cdot OO_2} = \frac{(R-12)^2+(R-15)^2-(3\sqrt{10})^2}{(R-12)(R-15)}.$$

Strategically subtracting 2 from both sides, we get

$$\frac{-18^2}{R^2} = \frac{-81}{(R-12)(R-15)} \implies R^2 = 4(R-12)(R-15) \implies R = 18 \pm 2\sqrt{21}.$$

Thus the answer is $\boxed{18+2\sqrt{21}}$ since the other root is extraneous (R needs to be at least 9 or else it will not contain two points X_1, X_2 at distance 18 from each other).

OR

Solution. Denote by P and Q the intersection points of Γ and ℓ, with P, Y_1, Y_2, and Q appearing in that order. Let M denote the midpoint of the minor arc $\widehat{PQ}$. Note that by Archimedes' Lemma, X_1, Y_1, and M are collinear, as are X_2, Y_2, and M. Furthermore, angle chasing yields

$$\angle MY_1Q = \frac{\widehat{MQ}+\widehat{X_1P}}{2} = \frac{\widehat{MP}+\widehat{PX_1}}{2} = \angle MX_2X_1,$$

and so $\triangle MY_1Y_2 \sim \triangle MX_2X_1$ with ratio of similitude 2.

Now as in the first solution let R denote the radius of Γ. Then homothety yields $\frac{X_1Y_1}{X_1M} = \frac{12}{R}$ and $\frac{X_2Y_2}{X_2M} = \frac{15}{R}$. As a result,

$$\frac{1}{4} = \left(\frac{MY_1}{MX_2}\right)\left(\frac{MY_2}{MX_1}\right) = \left(1-\frac{X_1Y_1}{X_1M}\right)\left(1-\frac{X_2Y_2}{X_2M}\right) = \frac{(R-12)(R-15)}{R^2}.$$

Solving this equation yields $R = \boxed{18+2\sqrt{21}}$ as before.

7. Let ABC be a triangle with $AB = 10$, $AC = 11$, and circumradius 6. Points D and E are located on the circumcircle of $\triangle ABC$ such that $\triangle ADE$ is equilateral. Line segments $\overline{DE}$ and $\overline{BC}$ intersect at X. Find $\frac{BX}{XC}$.

Proposed by David Altizio

Solution. Set $AX = d$, $BX = m$, $CX = n$. Let ω denote the circle centered at A with radius $AD = AE$. Then X lies on the radical axis of $\odot(ABC)$ and ω, and so the powers of X with respect to both circles are equal. In other words,

$$mn = BX \cdot XC = DX \cdot XE = AD^2 - d^2 = 3R^2 - d^2.$$

This rearranges to $mn + d^2 = 3R^2$, or, after multiplying by a,

$$3R^2a = amn + ad^2 = b^2m + c^2n,$$

where the last equality is an application of Stewart. Now substituting $a = m + n$ into the above equality yields

$$3R^2(m+n) = b^2m + c^2n \quad \Rightarrow \quad \frac{m}{n} = \frac{3R^2-c^2}{b^2-3R^2} = \boxed{\frac{8}{13}}.$$

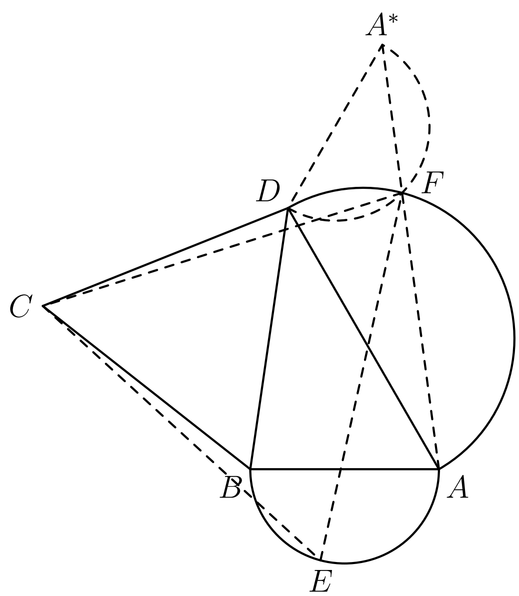
Remark. Essentially, we are applying Stewart's Theorem on two triangle/cevian pairs for which the value of $d^2 + mn$ is the same: the values of d^2 are identical trivially, while the values of mn are equal by Power of a Point.

8. In quadrilateral $ABCD$, $AB = 2$, $AD = 3$, $BC = CD = \sqrt{7}$, and $\angle DAB = 60^\circ$. Semicircles γ_1 and γ_2 are erected on the exterior of the quadrilateral with diameters $\overline{AB}$ and $\overline{AD}$; points $E \neq B$ and $F \neq D$ are selected on γ_1 and γ_2 respectively such that $\triangle CEF$ is equilateral. What is the area of $\triangle CEF$?

Proposed by Gunmay Handa

Solution. The following solution is intended to optimize the amount of arithmetic needed; more staight-forward solutions are possible.

First note that $\triangle CDB$ is equilateral, so there exists a spiral similarity between $\triangle CBD$ and $\triangle CEF$; this implies that $DF = EB$. Translate by vector $\vec{BD}$, and denote images with a $'$; we have that $\triangle DFE'$ is equilateral since $DF = DE'$ and $\angle(DF, DE') = \angle(DF, BE) = 60^\circ$. Now let Φ denote the transformation of the plane corresponding to rotation about C by 60°, as seen in the figure, and set $A^* = \Phi(A)$ for ease of typesetting. Then $\angle A^*DA = 120^\circ$ since Φ takes AB to A^*D and $\angle BAD = 60^\circ$. In addition, remark that $\triangle A^*FD \cong \triangle AEB$ since Φ takes the latter triangle to the former. This has many implications: F is the foot of the altitude from D to AA^*, $A^*F = AE$, and $\angle EAF = 120^\circ$. (Convince yourself that these properties hold!)



Now compute $AA^* = \sqrt{19}$ by Law of Cosines, so comparing areas gives $DF = \sqrt{\frac{27}{19}}$. Thus,

$$\begin{aligned} EF^2 &= AF^2 + AE^2 + AE \cdot AF = AF^2 + A^*F^2 + AF \cdot A^*F \\ &= (AF + A^*F)^2 - AF \cdot A^*F = (\sqrt{19})^2 - \sqrt{4 - \frac{27}{19}}\sqrt{9 - \frac{27}{19}} = 19 - \frac{7 \cdot 12}{19} = \frac{277}{19}. \end{aligned}$$

It follows easily that the area of $\triangle CEF$ is $\boxed{\frac{277\sqrt{3}}{76}}$.

9. Suppose $\mathcal{E}_1 \neq \mathcal{E}_2$ are two intersecting ellipses with a common focus X; let the common external tangents of $\mathcal{E}_1$ and $\mathcal{E}_2$ intersect at a point Y. Further suppose that X_1 and X_2 are the other foci of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, such that $X_1 \in \mathcal{E}_2$ and $X_2 \in \mathcal{E}_1$. If $X_1X_2 = 8, XX_2 = 7$, and $XX_1 = 9$, what is XY^2?

Proposed by Gunmay Handa

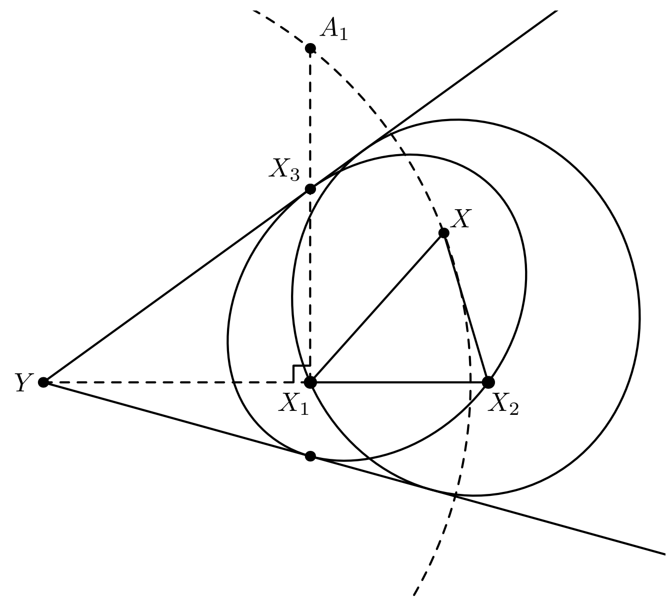
Solution. Our solution proceeds in two lemmas.

Lemma 1. *Suppose A_1 and A_2 are the reflections of X over each of the common external tangents. Then Y is the circumcenter of $\odot(A_1XA_2)$; moreover, $Y \in X_1X_2$.*

Proof. Let X_3 be the the tangency point of $\mathcal{E}_1$ nearer to A_1. Then

$$X_1A_1 = X_1X_3 + X_3A_1 = X_1X_3 + X_3X = 15$$

by the reflection property of ellipses. Hence we know that $\{A_1, A_2\}$ are the common points of the circles centered at X_1 and X_2 with radii 15 and 17, respectively. Finally, the common external tangents and the line X_1X_2 comprise the set of perpendicular bisectors of $\triangle XA_1A_2$; this implies the conclusion. □



Lemma 2. *The product of distances from the foci to a variable tangent of a fixed ellipse is constant.*

Proof. Given an ellipse with foci F_1, F_2, let P_1 and P_2 be the projections of F_1 and F_2 onto an arbitrary tangent to the ellipse at the point X. Let M be the midpoint of $\overline{F_1F_2}$; the reflection of P_2 about M produces a point P_3 with $F_2P_2 = F_1P_3$. In addition, we know that P_1, P_2 lie on a circle centered at M with radius $\frac{F_1X+F_2X}{2}$ by the reflection property and consequent dilations at F_2 and F_1 with ratio $\frac{1}{2}$, respectively. Hence, $P_1F_1 \cdot P_2F_2 = P_1F_1 \cdot P_3F_1$ is fixed at $(\frac{F_1X+F_2X}{2})^2 - (\frac{F_1F_2}{2})^2$ by Power of a Point on this circle. □

Let ℓ_1 be a common external tangent of the two ellipses. Then

$$\frac{YX_1}{YX_2} = \frac{d(X_1,\ell_1)}{d(X_2,\ell_1)} = \frac{d(X_1,\ell_1)d(X,\ell_1)}{d(X_2,\ell_1)d(X,\ell_1)} = \frac{15^2-9^2}{17^2-7^2} = \frac{3}{5}.$$

Since $X_1X_2 = 8$, we know $YX_1 = 12$. Note that $\triangle A_1X_1X_2$ is right, so

$$A_1Y^2 = A_1X_1^2 + X_1Y^2 = 15^2 + 12^2 = 369$$

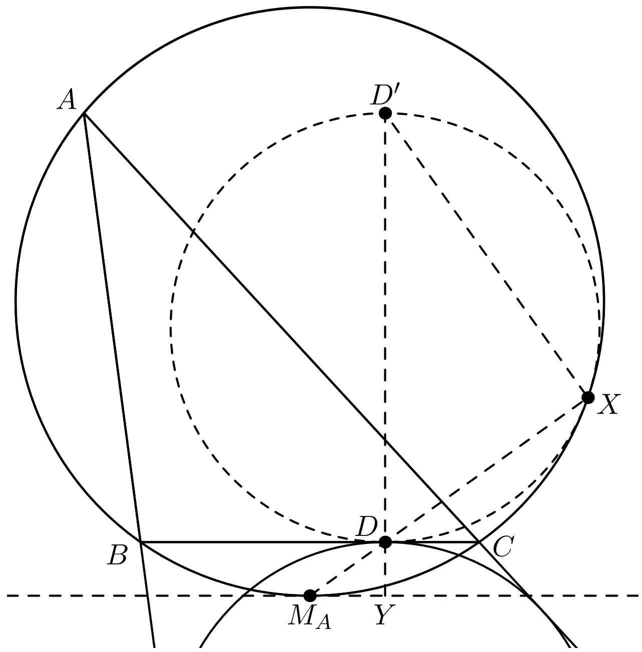
and since Y is the circumcenter of $\odot(A_1XA_2)$ we know $XY^2 = A_1Y^2 = \boxed{369}$, as desired (alternatively, one could use Stewart's theorem).

10. Let ABC be a triangle with circumradius 17, inradius 4, circumcircle Γ and A-excircle Ω. Suppose the reflection of Ω over line BC is internally tangent to Γ. Compute the area of $\triangle ABC$.

Proposed by Gunmay Handa

Solution. In this solution, define $BC = a$ and cyclic variants, and let K, R, s, r, r_a be the area, circumradius, semiperimeter, inradius and A-exradius of $\triangle ABC$, respectively.

Denote Ω' as the reflection of Ω over BC. Let D be the tangency point of Ω' on $\overline{BC}$, and suppose X is the tangency point of Ω' on Γ; note by Archimedes' lemma that DX passes through the midpoint M_A of minor arc BC in Γ. Let ℓ be the tangent to Γ at M_A, D' be the antipode of D in Ω', and $Y \equiv DD' \cap \ell$. From $\triangle DM_AY \sim \triangle DD'X$ we know $DY \cdot DD' = DM_A \cdot DX = DB \cdot DC$. Hence $DD' \cdot DY = 2r_a \cdot DY = (s-b)(s-c)$ and from the well-known identity $(s-b)(s-c) = rr_a$ we get that $DY = \frac{r}{2}$. Finally, if M is the midpoint of $\overline{BC}$, then $MM_A = DY$ and so Power of a Point at M gives $MB \cdot MC = \frac{r}{2}\left(2R - \frac{r}{2}\right) = 64$, whence $a = 16$.



By well-known formulas, we have

$$K = 4\left(\frac{16+b+c}{2}\right) = \frac{16bc}{4 \cdot 17}$$

so that $b + c = \frac{K-32}{2}$ and $bc = \frac{17K}{4}$. Then by Heron's formula,

$$\begin{aligned} 16K^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\ &= \frac{2K}{r}\left(-a + \frac{K-32}{2}\right)(a^2 - (b-c)^2) \\ &= \frac{K}{2}\left(\frac{K-64}{2}\right)(256 - (b+c)^2 + 4bc) \\ &= \frac{K}{2}\left(\frac{K-64}{2}\right)\left(33K - \frac{K^2}{4}\right) \end{aligned}$$

so $256 = (K-64)(132-K) \implies K = 68, 128$. It is easy to check that only 128 yields real b and c, so the answer is $\boxed{128}$.