CMIMC 2018
Geometry Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to the registration desk on the first floor of the University Center by the end of lunch.

Geometry

1. Let ABC be a triangle. Point P lies in the interior of $\triangle ABC$ such that $\angle ABP = 20^\circ$ and $\angle ACP = 15^\circ$. Compute $\angle BPC - \angle BAC$.

2. Let $ABCD$ be a square of side length 1, and let P be a variable point on $\overline{CD}$. Denote by Q the intersection point of the angle bisector of $\angle APB$ with $\overline{AB}$. The set of possible locations for Q as P varies along $\overline{CD}$ is a line segment; what is the length of this segment?

3. Let ABC be a triangle with side lengths 5, $4\sqrt{2}$, and 7. What is the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$?

4. Suppose $\overline{AB}$ is a segment of unit length in the plane. Let $f(X)$ and $g(X)$ be functions of the plane such that f corresponds to rotation about A 60° counterclockwise and g corresponds to rotation about B 90° clockwise. Let P be a point with $g(f(P)) = P$; what is the sum of all possible distances from P to line AB?

5. Select points T_1, T_2 and T_3 in $\mathbb{R}^3$ such that $T_1 = (0,1,0)$, T_2 is at the origin, and $T_3 = (1,0,0)$. Let T_0 be a point on the line $x = y = 0$ with $T_0 \neq T_2$. Suppose there exists a point X in the plane of $\triangle T_1T_2T_3$ such that the quantity $(XT_i)[T_{i+1}T_{i+2}T_{i+3}]$ is constant for all $i = 0$ to $i = 3$, where $[\mathcal{P}]$ denotes area of the polygon $\mathcal{P}$ and indices are taken modulo 4. What is the magnitude of the z-coordinate of T_0?

6. Let ω_1 and ω_2 be intersecting circles in the plane with radii 12 and 15, respectively. Suppose Γ is a circle such that ω_1 and ω_2 are internally tangent to Γ at X_1 and X_2, respectively. Similarly, ℓ is a line that is tangent to ω_1 and ω_2 at Y_1 and Y_2, respectively. If $X_1X_2 = 18$ and $Y_1Y_2 = 9$, what is the radius of Γ?

7. Let ABC be a triangle with $AB = 10$, $AC = 11$, and circumradius 6. Points D and E are located on the circumcircle of $\triangle ABC$ such that $\triangle ADE$ is equilateral. Line segments $\overline{DE}$ and $\overline{BC}$ intersect at X. Find $\frac{BX}{XC}$.

8. In quadrilateral $ABCD$, $AB = 2$, $AD = 3$, $BC = CD = \sqrt{7}$, and $\angle DAB = 60^\circ$. Semicircles γ_1 and γ_2 are erected on the exterior of the quadrilateral with diameters $\overline{AB}$ and $\overline{AD}$; points $E \neq B$ and $F \neq D$ are selected on γ_1 and γ_2 respectively such that $\triangle CEF$ is equilateral. What is the area of $\triangle CEF$?

9. Suppose $\mathcal{E}_1 \neq \mathcal{E}_2$ are two intersecting ellipses with a common focus X; let the common external tangents of $\mathcal{E}_1$ and $\mathcal{E}_2$ intersect at a point Y. Further suppose that X_1 and X_2 are the other foci of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, such that $X_1 \in \mathcal{E}_2$ and $X_2 \in \mathcal{E}_1$. If $X_1X_2 = 8$, $XX_2 = 7$, and $XX_1 = 9$, what is XY^2?

10. Let ABC be a triangle with circumradius 17, inradius 4, circumcircle Γ and A-excircle Ω. Suppose the reflection of Ω over line BC is internally tangent to Γ. Compute the area of $\triangle ABC$.