Algebra Individual Finals Solutions

1. For all real numbers r, denote by $\{r\}$ the fractional part of r, i.e. the unique real number $s \in [0, 1)$ such that $r - s$ is an integer. How many real numbers $x \in [1, 2)$ satisfy the equation $\left\{x^{2018}\right\} = \left\{x^{2017}\right\}$?

   Proposed by David Altizio

   Solution. The condition is equivalent to $x^{2018} = x^{2017} + N$ for some integer $N \geq 0$ (since $x^{2018} \geq x^{2017}$ due to $x \geq 1$). Note that the function $f(x) := x^{2018} - x^{2017}$ is continuous and increasing in $[1, 2)$, with $f(1) = 0$ and $f(2) = 2^{2017}$. Thus, for every $N \in \{0, \ldots, 2^{2017} - 1\}$, there exists exactly one $x_N \in [1, 2]$ for which $x_N^{2018} = x_N^{2017} + N$. Thus the requested answer is $\boxed{2^{2017}}$.

2. Compute the sum of the digits of

$$\prod_{n=0}^{2018}\left(10^{2\cdot 3^n} - 10^{3^n} + 1\right)\left(10^{2\cdot 3^n} + 10^{3^n} + 1\right).$$

   Proposed by Cody Johnson

   Solution. Let $a_n = 10^{3^n}$ for notational convienence. Note that $a_{n+1} = a_n^3$, so

$$\prod_{n=0}^{2018}(a_n^2 - a_n + 1)(a_n^2 + a_n + 1) = \prod_{n=0}^{2018}\frac{a_n^6 - 1}{a_n^2 - 1} = \frac{\prod_{n=0}^{2018}(a_{n+1}^2 - 1)}{\prod_{n=0}^{2018}(a_n^2 - 1)} = \frac{a_{2019}^2 - 1}{a_0^2 - 1}.$$

   Now $a_0^2 - 1 = 99$ and $a_{2019}^2 - 1$ is just a number consisting of $2 \cdot 3^{2019}$ nines, and so the answer is $\boxed{3^{2019}}$.

3. Let a be a complex number, and set α, β, and γ to be the roots of the polynomial $x^3 - x^2 + ax - 1$. Suppose

$$(\alpha^3 + 1)(\beta^3 + 1)(\gamma^3 + 1) = 2018.$$

   Compute the product of all possible values of a.

   Proposed by Keerthana Gurushankar and David Altizio

   Solution. Let ω be a primitive sixth root of unity, so that $\omega^3 = -1$ and $\omega^2 - \omega + 1 = 0$. Note that the product on the LHS becomes

$$-\prod_{cyc}(\alpha + 1)(\alpha - \omega)(\alpha - \bar{\omega}) = -p(-1)p(\omega)p(\bar{\omega}).$$

   It is not hard to compute $p(-1) = -3 - a$. Moreover, note that

$$p(\omega) = \omega^3 - (\omega^2 + 1) + a\omega = -1 + \omega(a - 1),$$

   and similarly $p(\bar{\omega}) = -1 + \bar{\omega}(a - 1)$. As a result,

$$p(\omega)p(\bar{\omega}) = (-1 + \omega(a - 1))(-1 + \bar{\omega}(a - 1)) = 1 - (a - 1) + (a - 1)^2 = a^2 - 3a + 3.$$

   So the given equation becomes

$$(a + 3)(a^2 - 3a + 3) = 2018$$

   and the product of the roots is $2018 - 9 = \boxed{2009}$.

Combinatorics Individual Finals Solutions

1. How many nonnegative integers with at most 40 digits consisting of entirely zeroes and ones are divisible by 11?

   Proposed by David Altizio

   Solution. Let $\omega := \exp(2\pi i/11)$, and consider the function

$$f(z) := (1+z)(1+z^{10})(1+z^{10^2})\dots(1+z^{10^{39}})$$

   Using a roots of unity filter, we seek

$$\frac{1}{11}\left[f(1)+f(\omega)+f(\omega^2)+\cdots+f(\omega^{10})\right]$$

   Note that

$$f(\omega^k)=(1+\omega^k)^{20}(1+\omega^{-k})^{20}=\omega^{-20k}(1+\omega^k)^{40}=\sum_{j=0}^{40}\binom{40}{j}\omega^{-20k+jk}=\sum_{j=-20}^{20}\binom{40}{j+20}\omega^{jk}$$

   Thus, the answer is

$$\frac{1}{11}\sum_{k=0}^{10}\sum_{j=-20}^{20}\binom{40}{j+20}\omega^{jk}=\sum_{j=-20}^{20}\left[\binom{40}{j+20}\sum_{k=0}^{10}\frac{1}{11}\omega^{jk}\right]=\boxed{2\binom{40}{9}+\binom{40}{20}}.$$

2. John has a standard four-sided die. Each roll, he gains points equal to the value of the roll multiplied by the number of times he has now rolled that number; for example, if he first rolls were $3, 3, 2, 3$, he would have $3+6+2+9=20$ points. Find the expected number of points he'll have after rolling the die 25 times.

   Proposed by Patrick Lin

   Solution. For general n, suppose instead that the multiplier is always decreased by one; we look at the number of times we've previously rolled a number. Then in total, the number of points we get from rolling 4's is equal to 4 times the number of pairs of rolls such that both rolls in the pair came out to be 4. In this scenario, the expected number of points would then just be $\binom{n}{2}\cdot\frac{1}{4}\cdot\frac{5}{2}$, since each pair has a $\frac{1}{4}$ chance to contribute (in expectancy) $\frac{5}{2}$ points.

   Now, when we add one to every multiplier this just increases the expected number of points by $\frac{5}{2}n$, so the answer is $\frac{5}{8}\binom{n}{2}+\frac{5}{2}n=\frac{5}{2}n\left(1+\frac{n-1}{8}\right)$. Substituting $n=25$ yields an answer of $\boxed{250}$.

3. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \dots, 2017\}$ with the following property: if S_1 and S_2 are two elements of $\mathcal{F}$ with $S_1 \subsetneq S_2$, then $|S_2 \setminus S_1|$ is odd. Compute the largest number of subsets $\mathcal{F}$ may contain.

   Proposed by David Altizio

   Solution. We claim the answer is $\boxed{\binom{2018}{1009}}$.

   First, we show that $|\mathcal{F}| \geq \binom{2018}{1009}+1 = 2\binom{2017}{1008}+1$ is impossible. Assume for the sake of contradiction there exists such a family $\mathcal{F}$ with at least that many elements, and set $N = |\mathcal{F}|$. Consider the poset $(\mathcal{F}, \subseteq)$ of all subsets in $\mathcal{F}$ ordered by inclusion. Note that $\mathcal{F}$ is a subset of $2^{[2017]}$, meaning that the poset P can be embedded into the Boolean lattice $\mathcal{B}_{2017}$[1]. By Sperner, the maximum size of an antichain in this lattice is $\binom{2017}{1008}$, meaning that in turn the maximum size of an antichain in $\mathcal{F}$ is at most $\binom{2017}{1008}$. Thus, since

$$2\binom{2017}{1008}=\binom{2018}{1009}<N,$$

[1] i.e. the poset of all subsets of [2017] ordered by inclusion

Dilworth guarantees the existence of a chain of length 3 in $\mathcal{F}$. In other words, there exist S_1, S_2, S_3 in $\mathcal{F}$ such that $S_1 \subsetneq S_2 \subsetneq S_3$. But now we have a contradiction: if $|S_2 \setminus S_1|$ and $|S_3 \setminus S_2|$ are both odd, then

$$|S_3 \setminus S_1| = |S_3 \setminus S_2| + |S_2 \setminus S_1|$$

is even.

It remains to construct an example of a family $\mathcal{F}$ with $\binom{2018}{1009}$ elements. Take

$$\mathcal{F} = \{S \subseteq [2017] : |S| = 1008 \text{ or } |S| = 1009\}.$$

Note that this family has exactly $\binom{2018}{1009}$ elements. Furthermore, if $S_1 \subsetneq S_2$ for S_1 and S_2 in $\mathcal{F}$, then it must be the case that $S_2 = S_1 \cup \{a\}$ for some $1 \leq a \leq 2017$, and so $|S_2 \setminus S_1| = 1$. Thus, we have a valid construction, and so we are done.

Computer Science Individual Finals Solutions

1. Consider a connected graph G with vertex set $\{0, 1, 2, ..., 6\}$. Suppose there exist 3 vertices of distance 1 away from vertex 0, 2 vertices of distance 2 away from vertex 0, and 1 vertex of distance 3 away from vertex 0. How many such graphs satisfy this property?

   Proposed by Cody Johnson

   Solution. There are
$$\frac{n!}{a_1! \cdot a_2! \cdot ... \cdot a_m!}$$
ways to choose which vertices are at each distance from vertex 0. Then, for each vertex at distance d away from vertex 0 ($1 \leq d \leq m$), we need to choose some nonempty subset of the vertices of distance $d-1$ to connect them to. Therefore, if we let $a_0 = 1$, there are
$$(2^{a_0} - 1)^{a_1} \cdot (2^{a_1} - 1)^{a_2} \cdot ... \cdot (2^{a_{m-1}} - 1)^{a_m}$$
Finally, we can connect vertices at the same distance together arbitrarily. There are
$$2^{\binom{a_1}{2}} \cdot 2^{\binom{a_2}{2}} \cdot ... \cdot 2^{\binom{a_m}{2}}$$
In total, there are thus
$$n! \cdot \prod_{i=1}^{m} \frac{(2^{a_{i-1}} - 1)^{a_i} \cdot 2^{\binom{a_i}{2}}}{a_i!}$$
such graphs. Plugging in the numbers gives an answer of
$$\frac{6!(2-1)^3(2^3-1)^2(2^2-1)^1 2^3 2^1 2^0}{3!2!1!} = \boxed{141120}.$$

2. Determine the largest number of steps for $\gcd(k, 76)$ to terminate over all choices of $0 < k < 76$, using the following algorithm for gcd. Give your answer in the form (n, k) where n is the maximal number of steps and k is the k which achieves this. If multiple k work, submit the smallest one.

```
FUNCTION gcd(a, b):
    IF a = 0 RETURN b
    ELSE RETURN gcd(b mod a, a)
```

   Proposed by Misha Ivkov and Gunmay Handa

   Solution. We claim the answer is $\boxed{(8, 47)}$. Denote by $\gcd(a, b)$ the number of steps needed for gcd to finish given two inputs a, b. First notice that $\gcd(F_{n-1}, F_n)$ takes the most steps to finish over all $\gcd(a, b)$ for $a < b < F_n$ (easy to show by induction). Hence $\gcd(k, 76) < \gcd(55, 89) = 9$ is our first upper bound. Now we split into four cases.

   - **Case 1.** If $k \leq 34$, then $\gcd(k, 76) = 1 + \gcd(76 \bmod k, k) < 1 + \gcd(21, 34) = 8$ (We note that $76 \bmod 34 \neq 21$, so this is strict inequality).
   - **Case 2.** If $34 < k \leq 38$, then $\gcd(k, 76) = 1 + \gcd(76 - 2k, k) < 1 + \gcd(34, 55) = 9$. Notice that $76 - 2k < 8$, so $\gcd(76 - 2k, k) < 1 + \gcd(5, 8) = 5$. Putting everything together again gives $\gcd(k, 76) < 6$, as in case 1.
   - **Case 3.** If $38 < k \leq 55$, then $\gcd(k, 76) = 1 + \gcd(76 - k, k) < 1 + \gcd(34, 55) = 9$ and we have no way of improving further.
   - **Case 4.** If $55 < k$, then $\gcd(k, 76) = 1 + \gcd(76 - k, k)$. Since $76 - k < 21$, then $\gcd(76 - k, k) = 1 + \gcd(k \pmod{76 - k}, 76 - k) < 1 + \gcd(13, 21) = 7$, so $\gcd(k, 76) < 8$.

Hence we simply analyze the numbers $k \in (38, 55]$ which are relatively prime to 76 to get that 47 indeed gives 8 steps.

3. For $n \in \mathbb{N}$, let x be the solution of $x^x = n$. Find the asymptotics of x, i.e., express $x = \Theta(f(n))$ for some suitable explicit function of n.

Proposed by Cody Johnson

Solution. We claim

$$x = \boxed{\Theta\left(\frac{\ln n}{\ln \ln n}\right)}.$$

Indeed, we claim that $\frac{\ln n}{\ln \ln n}$ is a lower bound on x. Notice that

$$\begin{aligned}\frac{\ln n}{\ln \ln n} < x &\iff \frac{\ln n}{\ln \ln n} \ln\left(\frac{\ln n}{\ln \ln n}\right) < x \ln x = \ln n \\ &\iff \frac{\ln n}{\ln \ln n}(\ln \ln n - \ln \ln \ln n) < \ln n \\ &\iff \ln n - \frac{\ln n \ln \ln \ln n}{\ln \ln n} < \ln n,\end{aligned}$$

which is clearly true for any $n > e^e$, so this is a lower bound. We also claim that $\frac{2 \ln n}{\ln \ln n}$ is an upper bound on x. Observe that

$$\begin{aligned}x < \frac{2 \ln n}{\ln \ln n} &\iff \ln n = x \ln x < \frac{2 \ln n}{\ln \ln n} \ln\left(\frac{2 \ln n}{\ln \ln n}\right) \\ &\iff \ln n < \frac{2 \ln n}{\ln \ln n}(\ln 2 \ln n - \ln \ln \ln n).\end{aligned}$$

Since $\ln 2 \ln n > \ln \ln n$ we can write

$$\begin{aligned}\frac{2 \ln n}{\ln \ln n}(\ln 2 \ln n - \ln \ln \ln n) &> \frac{2 \ln n}{\ln 2 \ln n}(\ln 2 \ln n - \ln \ln \ln n) \\ &> 2 \ln n - \ln n \cdot \frac{2 \ln \ln \ln n}{\ln 2 \ln n} \\ &> \ln n\left(2 - \frac{\ln \ln \ln n}{\ln 2 \ln n}\right) \\ &> \ln n,\end{aligned}$$

for large enough values of n. Hence x is bounded on both sides by scalar multiples of $\frac{\ln n}{\ln \ln n}$, as desired.

Geometry Individual Finals Solutions

1. Let ABC be a triangle with $AB = 9$, $BC = 10$, $CA = 11$, and orthocenter H. Suppose point D is placed on $\overline{BC}$ such that $AH = HD$. Compute AD.

   Proposed by David Altizio

   Solution. Let A' be the reflection over $\overline{BC}$. Then $\triangle AHD \sim \triangle ADA'$ since both triangles are isosceles, and so

$$AD^2 = AH \cdot AA' = 2AH \cdot d(A, BC) = 2AB \cdot AC \cos A = 2 \cdot 9 \cdot 11 \cdot \frac{17}{33} = 102,$$

   whence $AD = \boxed{\sqrt{102}}$.

   OR

   Solution. Recall that four points A, B, C, and D satisfy $AC \perp BD$ if and only if $AD^2 - CD^2 = AB^2 - CB^2$. With this in mind, write

$$BH^2 - AH^2 = BH^2 - DH^2 \quad \Rightarrow \quad BC^2 - AC^2 = AB^2 - AD^2.$$

   Rearranging thus yields

$$AD = \sqrt{AB^2 + AC^2 - BC^2} = \sqrt{9^2 + 11^2 - 10^2} = \boxed{\sqrt{102}}.$$

2. Suppose $ABCD$ is a trapezoid with $AB \parallel CD$ and $AB \perp BC$. Let X be a point on segment $\overline{AD}$ such that AD bisects $\angle BXC$ externally, and denote Y as the intersection of AC and BD. If $AB = 10$ and $CD = 15$, compute the maximum possible value of XY.

   Proposed by Gunmay Handa

   Solution. Let $Z = CX \cap AB$. Then XZ bisects $\angle BXZ$ from the definition of X, and so

$$\frac{BX}{BA} = \frac{ZX}{ZA} = \frac{XC}{CD} \quad \Rightarrow \quad \frac{BC}{XC} = \frac{AB}{CD} = \frac{2}{3}.$$

   Now let C' denote the reflection of C over CA. Then B, X, and C' are collinear with $\frac{BX}{XC'} = \frac{2}{3}$. But note that $\frac{BY}{YD} = \frac{2}{3}$ as well, and so $XY \parallel DC'$. In turn, when combined with $C'D = CD = 15$, we obtain

$$\frac{XY}{DC'} = \frac{XB}{C'B} = \frac{2}{5} \quad \Rightarrow \quad XY = \frac{2}{5} \cdot 15 = 6.$$

3. Let ABC be a triangle with incircle ω and incenter I. The circle ω is tangent to BC, CA, and AB at D, E, and F respectively. Point P is the foot of the angle bisector from A to BC, and point Q is the foot of the altitude from D to EF. Suppose $AI = 7$, $IP = 5$, and $DQ = 4$. Compute the radius of ω.

   Solution. We claim that in general

$$DQ = \frac{AP \cdot r^2}{AI \cdot IP},$$

   from which computation gives the answer as $\boxed{\frac{\sqrt{105}}{3}}$. There are many ways to prove this; we now present three of them.

   Solution 1: Invert about ω, and denote inverses with a *. Recall that since AE and AF are tangents to ω, A^* is the intersection point of $AI \equiv AP$ and EF. In a similar vein, P^* is the projection of D onto AP. But now DQA^*P^* is a rectangle, and so

$$DQ = A^*P^* = \frac{AP \cdot r^2}{AI \cdot IP}$$

by the Inversion Distance Formula.

Solution 2: Let X denote the foot of the perpendicular from A to BC. Recall that EF and BC are the polars of A and D respectively with respect to ω, so by Salmon's Theorem,

$$\frac{DQ}{AX} = \frac{\text{dist}(D, EF)}{\text{dist}(A, BC)} = \frac{r}{IA}.$$

In turn,

$$DQ = \frac{AX \cdot r}{IA} = \frac{AP \cdot r^2}{AI \cdot IP},$$

where the last step uses $\triangle PDI \sim \triangle PXA$.

Solution 3: Let $\triangle I_A I_B I_C$ denote the excentral triangle in the usual fashion. Recall that $\triangle DEF \sim \triangle I_A I_B I_C$ with similarity ratio $r : 2R$, as $\odot(ABC)$ is the nine-point circle of $\odot(I_A I_B I_C)$. Then

$$DQ = \frac{AI_A \cdot r}{2R} = \frac{AB \cdot AC \cdot r}{2R \cdot AI} = \frac{d(A, BC) \cdot r}{AI} = \frac{d(A, BC) \cdot r^2}{d(I, BC) \cdot AI} = \frac{AP \cdot r^2}{IP \cdot AI}.$$

Number Theory Individual Finals Solutions

1. Alex has one-pound red bricks and two-pound blue bricks, and has 360 total pounds of brick. He observes that it is impossible to rearrange the bricks into piles that all weigh three pounds, but he can put them in piles that each weigh five pounds. Finally, when he tries to put them into piles that all have three bricks, he has one left over. If Alex has r red bricks, find the number of values r could take on.

   Proposed by Patrick Lin

   Solution. Let b denote the number of blue bricks that Alex has. The first condition tells us that $r+2b = 360$. In particular, $r + 2b \equiv 0 \pmod 3$. The last condition tells us that $r + b \equiv 1 \pmod 3$. Thus, $b \equiv 2 \pmod 3$ and $r \equiv 2 \pmod 3$ also. The third condition informs us that the $r + b$ bricks can be partitioned into 72 piles that each weight 5 pounds. Since this cannot be done with only the two-pound bricks, there must be at least one one-pound brick in each pile; that is, $r \geq 72$.

   Now, the last condition informs us that b must be greater than 120. If there were fewer than 120 two-pound blue bricks, then we could form 120 piles, each with at most one two-pound blue brick, and then use the rest of the red one-pound bricks. Thus, $b > 120$ and since $r + 2b = 360$, $r < 120$.

   Thus, the conditions on r are $72 \leq r < 120$ and $r \equiv 2 \pmod 3$; it is not hard to calculate that there are $\boxed{16}$ possible such values of r.

2. How many integer values of k, with $1 \leq k \leq 70$, are such that $x^k - 1 \equiv 0 \pmod{71}$ has at least $\sqrt{k}$ solutions?

   Proposed by Andrew Kwon

   Solution. We use the well-known fact that $x^d - 1 \equiv 0 \pmod p$ has exactly d solutions when $d \mid p - 1$. It is also evident that if d is coprime to $p - 1$, then there is only the trivial solution.

   Now, any prime factors of k that do not divide $p - 1$ are irrelevant. In particular, the number of solutions to $x^k - 1 \equiv 0 \pmod p$ is exactly $\gcd(k, p-1)$. Therefore, we need $\gcd(k, p-1) \geq \sqrt{k}$, and this can now be counted manually by changing variables to $d = \gcd(k, p - 1)$, $k = d\ell$, where $\ell \leq \min(d, \frac{p-1}{d})$ and $\gcd(\ell, \frac{p-1}{d}) = 1$.

   - For $d = 1, 35, 70$ it is clear there is only one choice of ℓ and for $d = 2$, ℓ can be 1 or 2.
   - For $d = 5, 7$ there are 3 choices for ℓ, which are $\{1, 3, 5\}$ and $\{1, 3, 7\}$ respectively.
   - For $d = 10$, ℓ can be any positive integer at most 6; for $d = 14$, ℓ can be any positive integer at most 4.

   In total we find $\boxed{21}$ possible (d, ℓ) pairs which correspond to the desired values of k.

3. Determine the number of integers a with $1 \leq a \leq 1007$ and the property that both a and $a + 1$ are quadratic residues mod 1009.

   Proposed by Gunmay Handa

   Solution. Let $p = 1009$ be a general prime congruent to $1 \pmod 4$ and $(\frac{\cdot}{p})$ denote the Legendre symbol; this is the value of the sum

$$\begin{aligned}\frac{1}{4}\sum_{a=1}^{p-2}\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right) &= \frac{p-2}{4} + \frac{1}{4}\sum_{a=1}^{p-2}\left(\frac{a}{p}\right) + \frac{1}{4}\sum_{a=1}^{p-2}\left(\frac{a+1}{p}\right) + \frac{1}{4}\sum_{a=1}^{p-2}\left(\frac{a^2+a}{p}\right)\\ &= \frac{p-2}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}\sum_{a=1}^{p-2}\left(\frac{a^2}{p}\right)\left(\frac{1+a^{-1}}{p}\right)\\ &= \frac{p-4}{4} + \frac{1}{4}\sum_{b=2}^{p-2}\left(\frac{b}{p}\right) = \frac{p-5}{4}.\end{aligned}$$

   Thus our final answer is $\frac{p-5}{4} = \boxed{251}$.