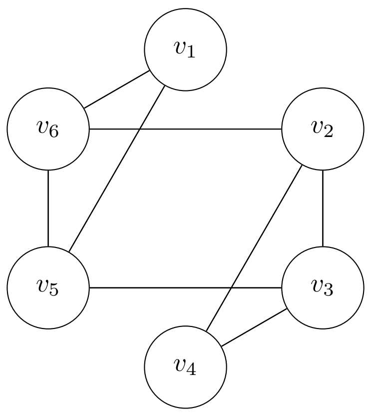
CMIMC 2018
Computer Science Round

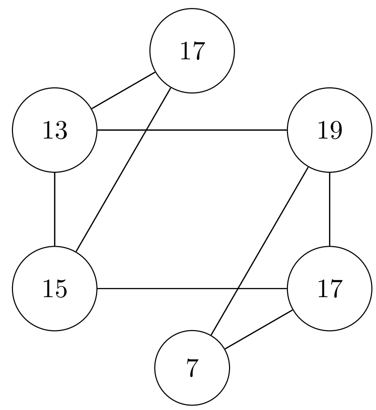
INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to the registration desk on the first floor of the University Center by the end of lunch.

Computer Science

1. Consider the following two vertex-weighted graphs, and denote them as having vertex sets $V = \{v_1, v_2, \ldots, v_6\}$ and $W = \{w_1, w_2, \ldots, w_6\}$, respectively (numbered in the same direction and way). The weights in the second graph are such that for all $1 \leq i \leq 6$, the weight of w_i is the sum of the weights of the neighbors of v_i. Determine the sum of the weights of the original graph.





2. Consider the natural implementation of computing Fibonacci numbers:

   FUNCTION FIB(n):
       IF $n = 0$ **OR** $n = 1$ **RETURN** 1
       RETURN FIB($n-1$) + FIB($n-2$)

   When FIB(10) is evaluated, how many recursive calls to FIB occur?

3. You are given the existence of an unsorted sequence $a_1, \ldots, a_5$ of five distinct real numbers. The Erdos-Szekeres theorem states that there exists a subsequence of length 3 which is either strictly increasing or strictly decreasing. You do not have access to the a_i, but you do have an oracle which, when given two indexes $1 \leq i < j \leq 5$, will tell you whether $a_i < a_j$ or $a_i > a_j$. What is the minimum number of calls to the oracle needed in order to identify an ordered triple of integers (r, s, t) such that a_r, a_s, a_t is one such sequence?

4. Consider the grid of numbers shown below.

20	01	96	56	16
37	48	38	64	60
96	97	42	20	98
35	64	96	40	71
50	58	90	16	89

   Among all paths that start on the top row, move only left, right, and down, and end on the bottom row, what is the minimum sum of their entries?

5. An *access pattern* π is a permutation of $\{1, 2, \ldots, 50\}$ describing the order in which some 50 memory addresses are accessed. We define the *locality* of π to be how much the program jumps around the memory, or numerically,

$$\sum_{i=2}^{50} |\pi(i) - \pi(i-1)| .$$

   If π is a uniformly randomly chosen access pattern, what is the expected value of its locality?

6. For integer $n \geq 2$ and real $0 \leq p \leq 1$, define $\mathcal{W}_{n,p}$ to be the complete weighted undirected random graph with vertex set $\{1, 2, \ldots, n\}$: the edge (i, j) will have weight $\min(i, j)$ with probability p and weight $\max(i, j)$ otherwise. Let $\mathcal{L}_{n,p}$ denote the total weight of the minimum spanning tree of $\mathcal{W}_{n,p}$. Find the largest integer less than the expected value of $\mathcal{L}_{2018,1/2}$.

7. I give you a function **rand** that returns a number chosen uniformly at random from $[0, T]$ for some number T that you don't know. Your task is to approximate T. You do this by calling **rand** 100 times, recording the results as $X_1, X_2, \ldots, X_{100}$, and guessing

$$\hat{T} = \alpha \cdot \max\{X_1, X_2, \ldots, X_{100}\}$$

for some α. Which value of α ensures that $\mathbb{E}[\hat{T}] = T$?

8. We consider a simple model for balanced parenthesis checking. Let $\mathcal{R} = \{\texttt{(())} \rightarrow \texttt{A}, \texttt{(A)} \rightarrow \texttt{A}, \texttt{AA} \rightarrow \texttt{A}\}$ be a set of rules for phrase reduction. Ideally, any given phrase is balanced if and only if the model is able to reduce the phrase to `A` by some arbitrary sequence of rule applications. For example, to show `((()))` is balanced we can perform the following sequence of reductions.

$$\texttt{((()))} \rightarrow \texttt{(A)} \rightarrow \texttt{A} \qquad \checkmark$$

Unfortunately, the above set of rules $\mathcal{R}$ is not complete, since there exist parenthetical phrases which are balanced but which are not balanced according to $\mathcal{R}$. Determine the number of such phrases of length 14.

9. Consider the following modified algorithm for binary search, which we will call *weighted binary search*:

```
FUNCTION SEARCH(L, value)
    hi ← len(L) − 1
    lo ← 0
    WHILE hi ≥ lo
        guess ← ⌊w · lo + (1 − w) · hi⌋
        mid ← L[guess]
        IF mid > value
            hi ← guess − 1
        ELSE IF mid < value
            lo ← guess + 1
        ELSE
            RETURN guess
    RETURN -1 (not found)
```

Assume L is a list of the integers $\{1, 2, \ldots, 100\}$, in that order. Further assume that accessing the kth index of L costs $k+1$ tokens (e.g. $L[0]$ costs 1 token). Let S be the set of all $w \in [\frac{1}{2}, 1)$ which minimize the average cost when `value` is an integer selected at random in the range $[1, 50]$. Given that $S = \left(x, \frac{74}{99}\right]$, determine x.

10. Consider an undirected, connected graph G with vertex set $\{v_1, v_2, \ldots, v_6\}$. Starting at the vertex v_1, an ant uses a DFS algorithm to traverse through G under the condition that if there are multiple unvisited neighbors of some vertex, the ant chooses the v_i with smallest i. How many possible graphs G are there satisfying the following property: for each $1 \leq i \leq 6$, the vertex v_i is the i^{th} new vertex the ant traverses?