CMIMC 2018
Algebra Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to the registration desk on the first floor of the University Center by the end of lunch.

Algebra

1. Misha has accepted a job in the mines and will produce one ore each day. At the market, he is able to buy or sell one ore for \$3, buy or sell bundles of three wheat for \$12 each, or *sell* one wheat for one ore. His ultimate goal is to build a city, which requires three ore and two wheat. How many dollars must Misha begin with in order to build a city after three days of working?

2. Suppose $x > 1$ is a real number such that $x + \frac{1}{x} = \sqrt{22}$. What is $x^2 - \frac{1}{x^2}$?

3. Let $P(x) = x^2 + 4x + 1$. What is the product of all real solutions to the equation $P(P(x)) = 0$?

4. 2018 little ducklings numbered 1 through 2018 are standing in a line, with each holding a slip of paper with a nonnegative number on it; it is given that ducklings 1 and 2018 have the number zero. At some point, ducklings 2 through 2017 change their number to equal the average of the numbers of the ducklings to their left and right. Suppose the new numbers on the ducklings sum to 1000. What is the maximum possible sum of the original numbers on all 2018 slips?

5. Suppose a, b, and c are nonzero real numbers such that
$$bc + \frac{1}{a} = ca + \frac{2}{b} = ab + \frac{7}{c} = \frac{1}{a+b+c}.$$
Find $a + b + c$.

6. We call $\overline{a_n \dots a_2}$ the Fibonacci representation of a positive integer k if
$$k = \sum_{i=2}^{n} a_i F_i,$$
where $a_i \in \{0, 1\}$ for all i, $a_n = 1$, and F_i denotes the i^{th} Fibonacci number ($F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for all $i \geq 2$). This representation is said to be *minimal* if it has fewer 1s than any other Fibonacci representation of k. Find the smallest positive integer that has eight ones in its minimal Fibonacci representation.

7. Compute
$$\sum_{k=0}^{2017} \frac{5 + \cos\left(\frac{\pi k}{1009}\right)}{26 + 10\cos\left(\frac{\pi k}{1009}\right)}.$$

8. Suppose P is a cubic polynomial satisfying $P(0) = 3$ and
$$(x^3 - 2x + 1 - P(x))(2x^3 - 5x^2 + 4 - P(x)) \leq 0$$
for all $x \in \mathbb{R}$. Determine all possible values of $P(-1)$.

9. Suppose $a_0, a_1, \dots, a_{2018}$ are integers such that
$$(x^2 - 3x + 1)^{1009} = \sum_{k=0}^{2018} a_k x^k$$
for all real numbers x. Compute the remainder when $a_0^2 + a_1^2 + \cdots + a_{2018}^2$ is divided by 2017.

10. Define a sequence of polynomials $F_n(x)$ by $F_0(x) = 0, F_1(x) = x - 1$, and for $n \geq 1$,
$$F_{n+1}(x) = 2xF_n(x) - F_{n-1}(x) + 2F_1(x).$$
For each n, $F_n(x)$ can be written in the form
$$F_n(x) = c_n P_1(x) P_2(x) \cdots P_{g(n)}(x)$$
where c_n is a constant and $P_1(x), P_2(x) \cdots, P_{g(n)}(x)$ are non-constant polynomials with integer coefficients and $g(n)$ is as large as possible. For all $2 < n < 101$, let t be the minimum possible value of $g(n)$ in the above expression; for how many k in the specified range is $g(k) = t$?