Number Theory Solutions Packet

1. There exist two distinct positive integers, both of which are divisors of 10^{10}, with sum equal to 157. What are they?

   Proposed by David Altizio

   Solution. Suppose $157 = x + y$ for x and y divisors of 10^{10}. Note that one of x or y must be odd and hence a power of 5. Similarly, one of x or y must be not divisible by 5, and hence a power of 2. Thus $157 = 2^a + 5^b$ for some nonnegative integers a and b. Now the largest power of 5 smaller than 157 is 125, and testing a few cases we indeed find that $157 - 125 = 32$ is the only solution which works. Thus the two integers are $\boxed{125 \text{ and } 32}$.

2. Determine all possible values of $m + n$, where m and n are positive integers satisfying

$$\operatorname{lcm}(m, n) - \gcd(m, n) = 103.$$

   Proposed by David Altizio

   Solution. Recall that by definition the least common multiple of two numbers is a multiple of their gcd. Let $\operatorname{lcm}(m, n) = k \cdot \gcd(m, n)$ for some positive integer k. Then

$$\operatorname{lcm}(m, n) - \gcd(m, n) = k \cdot \gcd(m, n) - \gcd(m, n) = \gcd(m, n)(k - 1) = 103.$$

   Recall that 103 is prime, so either $\gcd(m, n) = 103$ and $k = 2$ or $\gcd(m, n) = 1$ and $k = 104$. In the former case, let $m = 103m_0$ and $n = 103n_0$. Then

$$\operatorname{lcm}(103m_0, 103n_0) = 103 \operatorname{lcm}(m_0, n_0) = 103 \cdot 2,$$

   so $\operatorname{lcm}(m_0, n_0) = 2$. Combining this with the fact that $m \neq n$ means that m_0 and n_0 must be 1 and 2 in some order, i.e. $\{m, n\} = \{103, 206\}$. In the latter case, write $104 = 2^3 \cdot 13$. Since $\gcd(m, n) = 1$, it follows that m and n must either be 1 and 104 or 8 and 13 in some order. Combining both of these cases yields that $m + n$ must be either $\boxed{21,\ 105,\ \text{or } 309}$.

3. For how many triples of positive integers (a, b, c) with $1 \le a, b, c \le 5$ is the quantity

$$(a + b)(a + c)(b + c)$$

   not divisible by 4?

   Proposed by David Altizio

   Solution. Note that since the sum of the three multiplicands is $(a + b) + (b + c) + (c + a) = 2(a + b + c)$, we know that at least one of $a + b$, $b + c$, or $c + a$ is even. Thus the product is always divisible by 2. In order for the product to not be divisible by 4, it must be the case that two of these quantities are odd and the third one is congruent to 2 modulo 4.

   WLOG suppose that $a + b$ and $a + c$ are odd and $b + c \equiv 2 \pmod 4$. Since $(a + b) - (a + c) = b - c$ is even, it follows that b and c are of the same parity. We now split into cases based on whether both are even or both are odd.

   - If both are even, then they cannot both be congruent modulo 4, or else their sum would be divisible by 4. It follows that b and c must be 4 and 2 in some order. Then a can be 1, 3, or 5; this gives a total of $2 \times 3 = 6$ possibilities in this case.
   - If both are odd, then they both must be congruent modulo 4, or else their sum would be $1 + 3 \equiv 0 \pmod 4$. This means they must be both either $1 \pmod 4$ or $3 \pmod 4$. Then a can be either 2 or 4, so there are a total of $2 \times (2^2 + 1) = 10$ possibilities in this case.

Multiplying by 3 from our WLOG above gives the final answer as $3(6+10) = \boxed{48}$.

4. Let a_1, a_2, a_3, a_4, a_5 be positive integers such that a_1, a_2, a_3 and a_3, a_4, a_5 are both geometric sequences and a_1, a_3, a_5 is an arithmetic sequence. If $a_3 = 1575$, find all possible values of $|a_4 - a_2|$.

Proposed by Patrick Lin

Solution. Write the terms as

$$(a_1, a_2, a_3, a_4, a_5) = \left(\frac{m^2}{n^2}a, \frac{m}{n}a, a, \frac{p}{q}a, \frac{p^2}{q^2}a\right),$$

where m/n and p/q are reduced fractions and $a = 1575$. Then arithmetic sequence gives

$$m^2q^2 + n^2p^2 = 2n^2q^2.$$

Since m and n are coprime, it follows that $q \mid n$. Similarly, $n \mid q$ and hence $n = q$. We can rewrite

$$m^2 + p^2 = 2n^2.$$

Because each term is an integer, we also have $n^2 \mid a$, and hence $n = 1, 3, 5, 15$, since $1575 = 3^2 \cdot 5^2 \cdot 7$. Assume that $m \le p$; then the only triples (m, p, n) that satisfy these conditions are

$$(m, p, n) = (1, 1, 1), (3, 3, 3), (5, 5, 5), (15, 15, 15), (1, 7, 5), (3, 21, 15).$$

The possible ratios $(m/n, p/n)$ are hence $(1, 1)$ and $(1/5, 7/5)$, and so

$$a_4 - a_2 = a\left(\frac{p}{n} - \frac{m}{n}\right) \in \left\{0 \cdot 1575, \frac{6}{5} \cdot 1575\right\} = \boxed{\{0, 1890\}}.$$

5. One can define the greatest common divisor of two positive rational numbers as follows: for a, b, c, and d positive integers with $\gcd(a, b) = \gcd(c, d) = 1$, write

$$\gcd\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\gcd(ad, bc)}{bd}.$$

For all positive integers K, let $f(K)$ denote the number of ordered pairs of positive rational numbers (m, n) wiht $m < 1$ and $n < 1$ such that

$$\gcd(m, n) = \frac{1}{K}.$$

What is $f(2017) - f(2016)$?

Proposed by David Altizio

Solution. First remark that the gcd condition can be dropped, since if c and d are scaled up by a factor of k, both $\gcd(ad, bc)$ and bd are scaled up by k, and so their effects cancel out.

Note that

$$\frac{e}{f}\gcd\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{e}{f} \cdot \frac{\gcd(ad, bc)}{bd} = \frac{\gcd(ead, ebc)}{bdf} = \gcd\left(\frac{ae}{bf}, \frac{ce}{df}\right).$$

Hence this definition of gcd is in fact multiplicative, and so it suffices to find pairs of rational numbers m' and n' such that $\gcd(m', n') = 1$.

Write $m' = \frac{a'}{b'}$ and $n' = \frac{c'}{d'}$. Then

$$\gcd\left(\frac{a'}{b'}, \frac{c'}{d'}\right) = 1 \iff \gcd(a'd', b'c') = b'd'.$$

Let $a'd' = Mb'd'$ and $b'c' = Nb'd'$ for some integers M and N with $\gcd(M, N) = 1$. This simplifies to $\frac{a'}{b'} = M$ and $\frac{c'}{d'} = N$. So in fact, m' and n' are actually relatively prime integers.

Hence $f(K)$ is equal to the number of pairs of positive integers (M, N) with $1 \le M < K$ and $1 \le N < K$ such that $\gcd(M, N) = 1$. This in turn means that $f(2017) - f(2016)$ equals the number of such pairs with either $M = 2016$ or $N = 2016$. If $M = 2016$, then N can be any one of the integers for which $\gcd(N, 2016) = 1$, of which there are $\varphi(2016)$ of them. Similarly, $N = 2016$ yields $\varphi(2016)$ more ordered pairs. There is no possibility for overcounting, and so the final answer is

$$2\varphi(2016) = \boxed{1152}.$$

6. Find the largest positive integer N satisfying the following properties:

- N is divisible by 7;
- Swapping the i^{th} and j^{th} digits of N (for any i and j with $i \neq j$) gives an integer which is *not* divisible by 7.

Proposed by David Altizio

Solution. Write

$$N = \overline{a_k a_{k-1} \cdots a_1 a_0} = \sum_{m=0}^{k} 10^m a_m.$$

Suppose digits a_i and a_j are swapped, whgere $0 \le i < j \le k$, to form a new integer N'. Then it is not hard to see that

$$N - N' = \left(10^j a_j + 10^i a_i\right) - \left(10^j a_i + 10^i a_j\right) = \left(10^j - 10^i\right)(a_j - a_i).$$

The condition given in the problem statement is thus equivalent to this difference not being divisible by 7 for all i and j.

If $a_j - a_i$ is divisible by 7, then $a_i \equiv a_j \pmod 7$. This in turn means that all digits must have different residues modulo 7.

If $10^j - 10^i$ is divisible by 7, then $10^{j-i} \equiv 1 \pmod 7$. Remark that $\operatorname{ord}_7(10) = 6$, meaning that it must be the case that $j - i \equiv 0 \pmod 6$. This means that any such N must have at most 6 digits; if this were not the case, then swapping a_0 and a_6 would produce a new integer divisible by 7, thus violating the given conditions.

In all other cases, the difference will not be divisible by 7. Hence it suffices to find the largest integer N with at most six digits such that $N \equiv 0 \pmod 7$ and that each of the digits of N has a different remainder when divided by 7. With this im mind, suppose $k = 5$, and write

$$N \equiv \sum_{m=0}^{5} 10^m a_m \equiv 5a_5 + 4a_4 + 6a_3 + 2a_2 + 3a_1 + a_0 \pmod 7.$$

In the interest of being greedy, set $a_5 = 9$, $a_4 = 8$, and $a_3 = 7$; note that conveniently 987 is divisible by 7, so the search for possible N is reduces to finding a_0, a_1, $a_2 \in \{3, 4, 5, 6\}$ such that

$$2a_2 + 3a_1 + a_0 \equiv 0 \pmod 7.$$

Note that by the Rearrangement Inequality $2a_2 + 3a_1 + a_0$ must be at least $2 \cdot 4 + 3 \cdot 3 + 5 = 22$ and at most $2 \cdot 5 + 3 \cdot 6 + 4 = 32$. Hence in fact it must be true that

$$2a_2 + 3a_1 + a_0 = 28.$$

The only solutions to this under the given constraints is $(a_2, a_1, a_0) = (3, 6, 4)$ and $(a_2, a_1, a_0) = (5, 4, 6)$, so the largest N must be $\boxed{987546}$.

7. The *arithmetic derivative* $D(n)$ of a positive integer n is defined via the following rules:

- $D(1) = 0$;
- $D(p) = 1$ for all primes p;

- $D(ab) = D(a)b + aD(b)$ for all positive integers a and b.

Find the sum of all positive integers n below 1000 satisfying $D(n) = n$.

Proposed by Varun Kambhampati

Solution. Let N be a positive integer such that $D(N) = N$. Recall that we can write

$$N = p_1^{a_1} \cdots p_k^{a_k}$$

for some sequence of primes $\{p_j\}_{j=1}^k$ and exponents $\{a_j\}_{j=1}^k$. We now prove a lemma which explains how to compute arbitrary arithmetic derivatives.

LEMMA: We have

$$D\left(p_1^{a_1} \cdots p_k^{a_k}\right) = p_1^{a_1} \cdots p_k^{a_k} \left(\frac{a_1}{p_1} + \cdots + \frac{a_k}{p_k}\right).$$

Proof. We first show that $D(p^j) = jp^{j-1}$ for p a prime; this proves the claim in the case of $j = 1$. Fortunately, this is not hard. Write

$$D(p^j) = p^{j-1}D(p) + pD(p^{j-1}) = p^{j-1} + pD(p^{j-1}).$$

Now the claim follows from a simple induction argument.

To prove the lemma, we induct on k. The base case of $k = 1$ follows from the above paragraph. Now assume the inductive hypothesis holds for some k, and write

$$\begin{aligned} D\left(p_1^{a_1} \cdots p_k^{a_k}\right) &= p_1^{a_1} \cdots p_{k-1}^{a_{k-1}} D\left(p_k^{a_k}\right) + p_k^{a_k} D\left(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}\right) \\ &= p_1^{a_1} \cdots p_{k-1}^{a_{k-1}} \left(a_k p_k^{a_k - 1}\right) + p_k^{a_k}\left(p_1^{a_1} \cdots p_{k-1}^{a_{k-1}}\right)\left(\frac{a_1}{p_1} + \cdots + \frac{a_{k-1}}{p_{k-1}}\right) \\ &= p_1^{a_1} \cdots p_k^{a_k} \left(\frac{a_1}{p_1} + \cdots + \frac{a_k}{p_k}\right). \end{aligned}$$

Hence by induction we're done.

□

Going back to the original problem, note that $D(N) = N$ implies that

$$p_1^{a_1} \cdots p_k^{a_k} \left(\frac{a_1}{p_1} + \cdots + \frac{a_k}{p_k}\right) = p_1^{a_1} \cdots p_k^{a_k} \quad \Longrightarrow \quad \frac{a_1}{p_1} + \cdots + \frac{a_k}{p_k} = 1.$$

Multiplying both sides by $p_1 \cdots p_k$ yields

$$\frac{a_1 p_1 \cdots p_k}{p_1} + \cdots + \frac{a_k p_1 \cdots p_k}{p_k} = p_1 \cdots p_k.$$

Now take both sides modulo p_1. All but the first term goes away and so we are left with

$$a_1 p_2 \cdots p_k \equiv 0 \pmod{p_1}.$$

Thus $p_1 \mid a_1$. However, since

$$\frac{a_1}{p_1} + \cdots + \frac{a_k}{p_k} = 1,$$

the ratio $\frac{a_1}{p_1}$ cannot exceed 1. Hence we in fact have equality, meaning that $p_1 = a_1$ and $a_j = 0$ for all $2 \le j \le k$. It follows that $N = p^p$ for prime p. Since $5^5 > 1000$, the answer is simply $2^2 + 3^3 = \boxed{31}$.

8. Let N be the number of ordered triples $(a,b,c) \in \{1,\ldots,2016\}^3$ such that $a^2+b^2+c^2 \equiv 0 \pmod{2017}$. What are the last three digits of N?

Proposed by Andrew Kwon

Solution. We first claim that there are 2017^2 solutions if we allow a,b,c to equal 0. Letting z be such that $z^2 \equiv -1 \pmod{2017}$ (which we know exists because 2017 is a prime congruent to $1 \pmod 4$), the given congruence is equivalent to

$$a^2 \equiv (cz)^2 - b^2 = (cz-b)(cz+b) \pmod{2017}.$$

If $cz - b \equiv 0$, then there is one choice for a and 2017 choices for b from which c is uniquely determined. Otherwise, we have 2016 choices for the value of $cz-b$ and 2017 choices for the value of a from which the values $cz+b, b, c$ are determined. Thus, overall there are 2017^2 triples $(a,b,c) \in \{0,\ldots,2016\}^3$ satisfying the condition.

We now use inclusion-exclusion to get the desired count. There are $2\cdot 2017 - 1$ triples where $a=0$, as all choices for b except 0 yield two choices for c. The same is true for triples where $b=0, c=0$. This leads to a count of $2017^2 - 6\cdot 2017 + 3$, while the triple $(0,0,0)$ has been added once and removed thrice from the count, so we add 2 to get

$$N = 2017^2 - 6\cdot 2017 + 5 = (2017-5)(2017-1),$$

and the last three digits of N are $2012\cdot 2016 \equiv \boxed{192} \pmod{1000}$.

9. Find the smallest prime p for which there exist positive integers a,b such that

$$a^2 + p^3 = b^4.$$

Proposed by Andrew Kwon

Solution. We rewrite the equation as $p^3 = (b^2-a)(b^2+a)$, and as $b^2+a > b^2-a$ we have two cases.

- $b^2+a = p^2, b^2-a = p$: In this case, $2b^2 = p(p+1)$, and noting that $p \neq 2$ we have $b^2 = p(\frac{p+1}{2})$, from which we find $p|b$. However, then the right hand side must have at least two factors of p, while $p|\frac{p+1}{2}$ is impossible. Thus there are no solutions in this case.
- $b^2+a = p^3, b^2-a = 1$: In this case, $2b^2 = (p+1)(p^2-p+1)$, and we note again that $p \neq 2$. Now, $b^2 = (\frac{p+1}{2})(p^2-p+1)$, and we have

$$\gcd(\tfrac{p+1}{2}, p^2-p+1) = \gcd(p+1, p^2-p+1) = \gcd(p+1, 3).$$

  We split into further cases.

  - $p \equiv 1 \pmod 3$: As $\gcd(\frac{p+1}{2}, p^2-p+1) = 1$, we must have that $\frac{p+1}{2}, p^2-p+1$ are each perfect squares (since they are relatively prime and their product is a perfect square). Letting $n^2 = \frac{p+1}{2}, m^2 = p^2-p+1$, we note that $n>1$ and so

$$(2n^2-2)^2 < (2n^2-1)^2 - (2n^2-1) + 1 = m^2 < (2n^2-1)^2,$$

    which is impossible. Once again, we find no solutions.
  - $p \equiv 2 \pmod 3$: By an argument similar to before, we must have $\frac{p+1}{2}, p^2-p+1$ are each 3 times a perfect square. Letting $3n^2 = \frac{p+1}{2}, 3m^2 = p^2-p+1$ we find $p = 6n^2-1, 3m^2 = p^2-p+1$. For $n=1$ we do have $p=5$, however there are no m such that $3m^2 = 21$. On the other hand, for $n=2$ we find $p = 23, p^2-p+1 = 507 = 3\cdot 169$, and so $\boxed{23}$ is the smallest valid value for p. Explicitly, $a = 6083, p = 23, b = 78$ is the complete solution to the original diophantine.

10. For each positive integer n, define

$$g(n) = \gcd\{0!n!, 1!(n-1)!, 2(n-2)!, \ldots, k!(n-k)!, \ldots, n!0!\}.$$

Find the sum of all $n \le 25$ for which $g(n) = g(n+1)$.

Proposed by Cody Johnson and Andrew Kwon

Solution. We claim $g(n) = \frac{(n+1)!}{\text{lcm}(1,\ldots,n+1)}$, and it suffices to show

$$\nu_p(g(n)) = \nu_p((n+1)!) - \nu_p(\text{lcm}(1,\ldots,n+1))$$

for each prime p. Noting that $k!(n-k)! = n!/\binom{n}{k}$, we have

$$\begin{aligned}\nu_p(g(n)) &= \min_{1\le k\le n}\left[\nu_p(n!) - \nu_p\left(\binom{n}{k}\right)\right]\\ &= \nu_p(n!) - \max_{1\le k\le n}\nu_p\left(\binom{n}{k}\right),\end{aligned}$$

and so

$$\begin{aligned}&\nu_p(g(n)) = \nu_p((n+1)!) - \nu_p(\text{lcm}(1,\ldots,n+1))\\ \Leftrightarrow\ &\nu_p(n!) - \max_{1\le k\le n}\nu_p\left(\binom{n}{k}\right) = \nu_p(n+1) + \nu_p(n!) - \max_{1\le j\le n+1}\nu_p(j)\\ \Leftrightarrow\ &\max_{1\le j\le n+1}\nu_p(j) = \nu_p(n+1) + \max_{1\le k\le n}\nu_p\left(\binom{n}{k}\right).\end{aligned}$$

Now, consider ℓ such that $p^\ell \le n+1 < p^{\ell+1}$, so that $\max_j \nu_p(j) = \ell$. It suffices to show that $\nu_p(n+1) + \max_k \nu_p(\binom{n}{k}) = \ell$. Suppose for the sake of contradiction that $p^{\ell+1} | (n+1)\binom{n}{k}$ for some k. Note that

$$(n+1)\binom{n}{k} = (k+1)\binom{n+1}{k+1} = (n-k+1)\binom{n+1}{k},$$

while evidently

$$\begin{aligned}\nu_p\left(\binom{n}{k}\right) &= \sum_{s=1}^{\ell}\left(\left\lfloor\frac{n}{p^s}\right\rfloor - \left\lfloor\frac{k}{p^s}\right\rfloor - \left\lfloor\frac{n-k}{p^s}\right\rfloor\right)\\ &\le \ell,\end{aligned}$$

and similarly for $\binom{n+1}{k+1}, \binom{n+1}{k}$. Thus, we must have $p \mid n+1, k+1, n-k+1$, implying that $p \mid (n-k+1) + (k+1) - (n+1) = 1$, which is impossible. To finish, we note that $k = p^\ell - 1$ yields $p^\ell \mid (k+1)\binom{n+1}{k+1}$.

We conclude that $g(n) = \frac{(n+1)!}{\text{lcm}(1,\ldots,n+1)}$, and now claim that $g(n) = g(n+1) \Leftrightarrow n+2$ is prime. Supposing

$$\frac{(n+1)!}{\text{lcm}(1,\ldots,n+1)} = \frac{(n+2)!}{\text{lcm}(1,\ldots,n+2)},$$

we have $n+2 = \frac{\text{lcm}(m,n+2)}{m}$, where $m = \text{lcm}(1,\ldots,n+1)$. Thus, $n+2$ is relatively prime to $1,\ldots,n+1$ and is prime. The other direction is clear.

Thus, the desired $n \le 25$ are $1, 3, 5, 9, 11, 15, 17, 21$ and their sum is $\boxed{82}$.