# CMIMC 2016

# Team Round

## INSTRUCTIONS

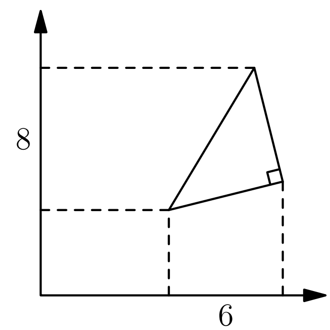
1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 30 minutes.
3. Write your team name and team ID on your answer sheet.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. All answers are integers.
7. If you believe that the test contains an error, submit your protest in writing to Porter 100.

# Team

1. Construction Mayhem University has been on a mission to expand and improve its campus! The university has recently adopted a new construction schedule where a new project begins every two days. Each project will take exactly one more day than the previous one to complete (so the first project takes 3, the second takes 4, and so on.)

   Suppose the new schedule starts on Day 1. On which day will there first be at least 10 projects in place at the same time?

2. Right isosceles triangle $T$ is placed in the first quadrant of the coordinate plane. Suppose that the projection of $T$ onto the $x$-axis has length 6, while the projection of $T$ onto the $y$-axis has length 8. What is the sum of all possible areas of the triangle $T$?




3. We have 7 buckets labelled 0-6. Initially bucket 0 is empty, while bucket $n$ (for each $1 \leq n \leq 6$) contains the list $[1, 2, \ldots, n]$. Consider the following program: choose a subset $S$ of $[1, 2, \ldots, 6]$ uniformly at random, and replace the contents of bucket $|S|$ with $S$. Let $\frac{p}{q}$ be the probability that bucket 5 still contains $[1, 2, \ldots, 5]$ after two executions of this program, where $p, q$ are positive coprime integers. Find $p$.

4. For some integer $n > 0$, a square paper of side length $2^n$ is repeatedly folded in half, right-to-left then bottom-to-top, until a square of side length 1 is formed. A hole is then drilled into the square at a point $\frac{3}{16}$ from the top and left edges, and then the paper is completely unfolded. The holes in the unfolded paper form a rectangular array of unevenly spaced points; when connected with horizontal and vertical line segments, these points form a grid of squares and rectangles. Let $P$ be a point chosen randomly from *inside* this grid. Find the largest $L$ such that, for all $n$, the probability that the four segments $P$ is bounded by form a square is at least $L$.

5. Recall that in any row of Pascal's Triangle, the first and last elements of the row are 1 and each other element in the row is the sum of the two elements above it from the previous row. With this in mind, define the *Pascal Squared Triangle* as follows:

   - In the $n^{\text{th}}$ row, where $n \geq 1$, the first and last elements of the row equal $n^2$;
   - Each other element is the sum of the two elements directly above it.

   The first few rows of the Pascal Squared Triangle are shown below.

   | | | | | | | | | | | |
   |---|---|---|---|---|---|---|---|---|---|---|
   | Row 1: | | | | | 1 | | | | | |
   | Row 2: | | | | 4 | | 4 | | | | |
   | Row 3: | | | 9 | | 8 | | 9 | | | |
   | Row 4: | | 16 | | 17 | | 17 | | 16 | | |
   | Row 5: | 25 | | 33 | | 34 | | 33 | | 25 | |

   Let $S_n$ denote the sum of the entries in the $n^{\text{th}}$ row. For how many integers $1 \leq n \leq 10^6$ is $S_n$ divisible by 13?

6. Suppose integers $a < b < c$ satisfy

$$a + b + c = 95 \qquad \text{and} \qquad a^2 + b^2 + c^2 = 3083.$$

Find $c$.

7. In $\triangle ABC$, $AB = 17$, $AC = 25$, and $BC = 28$. Points $M$ and $N$ are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively, and $P$ is a point on $\overline{BC}$. Let $Q$ be the second intersection point of the circumcircles of $\triangle BMP$ and $\triangle CNP$. It is known that as $P$ moves along $\overline{BC}$, line $PQ$ passes through some fixed point $X$. Compute the sum of the squares of the distances from $X$ to each of $A$, $B$, and $C$.

8. Let $N$ be the number of triples of positive integers $(a, b, c)$ with $a \le b \le c \le 100$ such that the polynomial

$$P(x) = x^2 + (a^2 + 4b^2 + c^2 + 1)x + (4ab + 4bc - 2ca)$$

has integer roots in $x$. Find the last three digits of $N$.

9. For how many permutations $\pi$ of $\{1, 2, \dots, 9\}$ does there exist an integer $N$ such that

$$N \equiv \pi(i) \pmod{i} \text{ for all integers } 1 \le i \le 9?$$

10. Let $\mathcal{P}$ be the unique parabola in the $xy$-plane which is tangent to the $x$-axis at $(5, 0)$ and to the $y$-axis at $(0, 12)$. We say a line $\ell$ is $\mathcal{P}$-friendly if the $x$-axis, $y$-axis, and $\mathcal{P}$ divide $\ell$ into three segments, each of which has equal length. If the sum of the slopes of all $\mathcal{P}$-friendly lines can be written in the form $-\frac{m}{n}$ for $m$ and $n$ positive relatively prime integers, find $m + n$.