Number Theory Solutions

1. David, when submitting a problem for CMIMC, wrote his answer as $100\frac{x}{y}$, where x and y are two positive integers with $x < y$. Andrew interpreted the expression as a product of two rational numbers, while Patrick interpreted the answer as a mixed fraction. In this case, Patrick's number was exactly double Andrew's! What is the smallest possible value of $x + y$?

   Proposed by David Altizio

   Solution. According to the problem statement, Andrew interpreted David's result as $\frac{100x}{y}$, while Patrick interpreted it as $100 + \frac{x}{y}$. Since Patrick's number was twice as large as Andrew's we have

$$\frac{200x}{y} = 100 + \frac{x}{y} \implies \frac{x}{y} = \frac{100}{199}.$$

   Therefore the smallest possible value of $x + y$ is $\boxed{299}$, achieved when $x = 100$ and $y = 199$.

2. Let a_1, a_2, … be an infinite sequence of integers such that k divides $\gcd(a_{k-1}, a_k)$ for all $k \geq 2$. Compute the smallest possible value of $a_1 + a_2 + \cdots + a_{10}$.

   Proposed by David Altizio

   Solution. Note that the condition implies that a_k is divisible by both $k, k+1$ for all $k \geq 1$. In particular, $a_k \geq k(k+1)$. Also, the construction $a_k = k(k+1)$ will satisfy the conditions of the problem, so the smallest possible value of the sum $a_1 + \ldots + a_{10}$ is

$$1 \cdot 2 + 2 \cdot 3 + \ldots + 10 \cdot 11 = \boxed{440}.$$

3. How many pairs of integers (a, b) are there such that $0 \leq a < b \leq 100$ and such that $\frac{2^b - 2^a}{2016}$ is an integer?

   Proposed by Cody Johnson

   Solution. Factoring 2016 as $2^5 \cdot 3^2 \cdot 7$, it follows that $2^5 | 2^b - 2^a$, whence $a \geq 5$, and also $9 | 2^b - 2^a$, whence $6 | b - a$. Consider $b - a = 6n$ for some positive integer n. Then, $5 \leq a \leq 100 - 6n$, and so there are $96 - 6n$ possible values of a with precisely one corresponding value of b for a given n. Note that $n > 0$ because $b > a$. Thus, the number of pairs can be counted by

$$\sum_{n=1}^{16} 96 - 6n = 96 \cdot 16 - 16 \left(\frac{16 \cdot 17}{2} \right)$$

   which evaluates to $\boxed{720}$.

4. For some positive integer n, consider the usual prime factorization

$$n = \prod_{i=1}^{k} p_i^{e_i} = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k},$$

   where k is the number of primes factors of n and p_i are the prime factors of n. Define $Q(n), R(n)$ by

$$Q(n) = \prod_{i=1}^{k} p_i^{p_i} \text{ and } R(n) = \prod_{i=1}^{k} e_i^{e_i}.$$

   For how many $1 \leq n \leq 70$ does $R(n)$ divide $Q(n)$?

Proposed by Andrew Kwon

Solution. I claim that, by counting the complement, only the n with $e_i \neq 1, p_i$ need be considered. Indeed, if $e_i = 1, p_i$ for all i then it is evident that $R(n)|Q(n)$. Now, we consider the multiples of 8,9,25, or 49 less than 70, since this is a superset of the possible n with some $e_i \neq 1, p_i$. Note that these are all disjoint.

- For multiples of 8, the multiples 8, 16, 32, 40, 48, 56, 64 fail, which contributes 7 failures.
- For multiples of 9, the multiples 9, 18, 36, 45, 54, 63 fail, which contributes 6 failures.
- For multiples of 25, the multiple 25, 50 fail, which contributes 2 failures.
- For multiples of 49, the multiple 49 fails, which contributes 1 failure.

These are the only integers which fail from 2 to 70, of which there are 16. The number 1 also works (an empty product by default evaluates to 1.) Thus, there are 69-16+1 = $\boxed{54}$ integers n such that $R(n)$ divides $Q(n)$.

5. Determine the sum of the positive integers n such that there exist primes p, q, r satisfying $p^n + q^2 = r^2$.

Proposed by Andrew Kwon

Solution. By parity, one of the primes must be 2, while $r \neq 2$.

First consider the case when $p = 2$. Then, $2^n = r^2 - q^2 = (r-q)(r+q)$, and so $r - q, r + q$ are powers of 2, say $2^a, 2^b$, with $0 \leq a < b$. Then, $r = \frac{1}{2}(2^a + 2^b)$. If $a = 0$, then r is not an integer; if $a > 1$, then $b > a > 1$, and r is even. Neither of these are possible, and so $a = 1$. Thus we can write $r = 2^{b-1}+1, q = 2^{b-1}-1$. Since $2^{b-1} \equiv \pm 1 \pmod 3$, it follows that one of r, q must be divisible by 3; $r = 3 \implies q = 1$, is impossible, and so we find that $q = 3, r = 5$ is a possible solution with $b = 3$. In this case we find that $n = 4$.

Otherwise, suppose $q = 2$. Then, $p^n = (r-2)(r+2)$, and as before we may write $r - 2 = p^a, r + 2 = p^b$. Then, $r = \frac{1}{2}(p^a + p^b)$ but $p \not| r \implies a = 0$. Now, $2 = \frac{1}{2}(p^b - 1)$, and so $p = 5, b = 1, r = 3$, and $n = 1$.

These are the only solutions, and so the sum of the possible n is $\boxed{5}$.

6. Define a *tasty residue* of n to be an integer $1 \leq a \leq n$ such that there exists an integer $m > 1$ satisfying

$$a^m \equiv a \pmod n.$$

Find the number of tasty residues of 2016.

Proposed by Andrew Kwon

Solution. The number of tasty residues of $n = p_i^{e_i} \cdots p_k^{e_k}$ is

$$\prod_{i=1}^{k} (\varphi(p_i^{e_i}) + 1).$$

Indeed, we need $p_i^{e_i} | a^m - a$ for some $m > 1$. For each of these relatively prime moduli, this can occur only in $\varphi(p_i^{e_i}) + 1$ ways; either a is relatively prime to p_i, or $p_i^{e_i} | a$. Thus, by the Chinese Remainder Theorem there are

$$\prod_{i=1}^{k} (\varphi(p_i^{e_i}) + 1)$$

total solutions modulo n. For $n = 2016$, this evaluates to $\boxed{833}$.

7. Determine the smallest positive prime p which satisfies the congruence

$$p + p^{-1} \equiv 25 \pmod{143}.$$

Here, p^{-1} as usual denotes multiplicative inverse.

Proposed by David Altizio

Solution. Multiply both sides of the equivalence by p to obtain $p^2 + 1 \equiv 25p \pmod{143}$. This means that

$$p^2 - 25p + 1 \equiv p^2 - 25p + 144 \equiv (p-9)(p-16) \equiv 0 \pmod{143}.$$

Note that $p = 9, 16$ are trivially solutions to this congruence, but there are other ones as well. In particular, note that $p - 9 \equiv 0 \pmod{11}$ and $p - 16 \equiv 0 \pmod{13}$ gives $p \equiv 42 \pmod{143}$, while $p - 9 \equiv 0 \pmod{13}$ and $p - 16 \equiv 0 \pmod{11}$ gives $p \equiv 126 \pmod{143}$.

Now the rest of the problem is straightforward. Remark that $9, 16, 42, 126$ are all composite, so we add 143 to each of these residues to get the next set of possible primes: 152, 159, 185, 269. The first three can be shown to be composite, while $\boxed{269}$ is prime, and the smallest possible prime satisfying these conditions.

8. Given that

$$\sum_{x=1}^{70} \sum_{y=1}^{70} \frac{x^y}{y} = \frac{m}{67!}$$

for some positive integer m, find $m \pmod{71}$.

Proposed by Andrew Kwon

Solution. Consider $\displaystyle\sum_{x=1}^{70} \frac{x^y}{y}$ for a fixed $y, 1 \le y \le 69$. Because 71 is prime, it has some primitive root, say r, and $\{1, r, \ldots, r^{69}\}$ is the set of all residues modulo 71. It follows that

$$\sum_{x=1}^{70} x^y \equiv \sum_{n=0}^{70} r^{ny} \pmod{71}.$$

However, the right hand side is a geometric series in r, which we evaluate to be $\dfrac{r^{71y}-1}{r^y - 1}$, where we formally treat division as multiplication by multiplicative inverses modulo 71; note that $y < 70 \implies r^y - 1 \not\equiv 0 \pmod{71}$, and so the above expression is well-defined modulo 71. Thus,

$$\sum_{x=1}^{70} x^y \equiv \frac{r^{71y}-1}{r^y - 1} \pmod{71},$$

while $r^{71y} - 1 \equiv 0 \pmod{71}$. Thus, for each $1 \le y \le 69$, the numerator of

$$\sum_{x=1}^{70} \frac{x^y}{y}$$

is divisible by 71. On the other hand, the case where $y = 70$ yields

$$\sum_{x=1}^{70} x^{70} \equiv 70 \pmod{71}.$$

Now, for each $1 \le y \le 69$, we have

$$67! \sum_{x=1}^{70} \frac{x^y}{y}$$

is an integer, and is divisible by 71. Thus, these terms do not contribute to $m \pmod{71}$. Finally, we consider

$$67!\sum_{x=1}^{70}\frac{x^{70}}{70},$$

which is also an integer, and so

$$\begin{aligned} m &\equiv 67! \cdot 70^{-1}\sum_{x=1}^{70} x^{70} \pmod{71} \\ &\equiv 67! \pmod{71}. \end{aligned}$$

Given Wilson's Theorem, it's evident that

$$\begin{aligned} m \cdot 68 \cdot 69 \cdot 70 &\equiv 70! \pmod{71} \\ \implies 6m &\equiv 1 \pmod{71}, \end{aligned}$$

and so $m \equiv \boxed{12} \pmod{71}$.

9. Compute the number of positive integers $n \leq 50$ such that there exist distinct positive integers a, b satisfying

$$\frac{a}{b} + \frac{b}{a} = n\left(\frac{1}{a} + \frac{1}{b}\right).$$

Proposed by David Altizio, solution by Andrew Kwon

Solution. Multiplying both sides of the equation by ab yields

$$a^2 + b^2 = n(a+b).$$

Now, $a^2 + b^2 \equiv 0 \pmod{a+b}$, and so $ab \equiv 0 \pmod{a+b}$, and also $a^2 \equiv 0 \pmod{a+b}$. Now, let $d = \gcd(a, b)$ so that $a = da', b = db'$, with a', b' relatively prime. Then, $a + b|a^2$ is equivalent to $a' + b'|d(a')^2$. However, $a' + b'$ cannot divide $(a')^2$, and thus $a' + b'|d$. Finally,

$$n = \frac{a^2+b^2}{a+b} = \frac{d}{a'+b'}((a')^2 + (b')^2).$$

In particular, $\dfrac{d}{a'+b'}$ must be an integer, and so it follows that n is any multiple of a sum of relatively prime squares. It is well-known that any prime dividing a sum of squares must be $1 \pmod 4$, and so n need only have a prime factor that is $1 \pmod 4$. The primes that satisfy this less than 50 are $5, 13, 17, 29, 37, 41$, and they contribute $10, 3, 2, 1, 1, 1$ possible n respectively. Thus, the total possible number of n is $\boxed{18}$.

10. Let $f : \mathbb{N} \mapsto \mathbb{R}$ be the function

$$f(n) = \sum_{k=1}^{\infty}\frac{1}{\operatorname{lcm}(k,n)^2}.$$

It is well-known that $f(1) = \frac{\pi^2}{6}$. What is the smallest positive integer m such that $m \cdot f(10)$ is the square of a rational multiple of π?

Proposed by Cody Johnson

Solution. For $d \in \{1, 2, 5, 10\}$, let $S_d := \displaystyle\sum_{\gcd(k,10)=d}\frac{1}{k^2}$ and $T_d := \displaystyle\sum_{d|k}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{(kd)^2} = \frac{\pi^2}{6d^2}$. Then we have

$$\sum_{k=1}^{\infty}\frac{1}{\operatorname{lcm}(k,10)^2} = \sum_{k=1}^{\infty}\frac{\gcd(k,10)^2}{k^2 \cdot 10^2} = \frac{1}{10^2}\left[1^2 \cdot S_1 + 2^2 \cdot S_2 + 5^2 \cdot S_5 + 10^2 \cdot S_{10}\right]$$

Note that $T_{10} = S_{10}$, $T_5 = S_5 + S_{10} = S_5 + T_{10}$, $T_2 = S_2 + S_{10} = S_2 + T_{10}$, and $T_1 = S_1 + S_2 + S_5 + S_{10} = S_1 + T_2 + T_5 - T_{10}$. Therefore, the sum evaluates to

$$\frac{1}{10^2}\left[1^2 \cdot (T_1 - T_2 - T_5 + T_{10}) + 2^2 \cdot (T_2 - T_{10}) + 5^2 \cdot (T_5 - T_{10}) + 10^2 \cdot T_{10}\right] = \frac{343\pi^2}{60000} = \frac{7^3\pi^2}{6 \cdot 10^4}$$

Thus, $m = 6 \cdot 7 = \boxed{42}$.