Geometry Solutions Packet

1. Let $\triangle ABC$ be an equilateral triangle and P a point on $\overline{BC}$. If $PB = 50$ and $PC = 30$, compute PA.

   Proposed by David Altizio

   Solution. Let M be the midpoint of $\overline{BC}$. The fact that $PB = 50$ and $PC = 30$ implies that the side length of $\triangle ABC$ is 80, so $AM = 40\sqrt{3}$. Furthermore, it is easy to deduce that $BM = 10$. Therefore by the Pythagorean Theorem

   $$AP^2 = AM^2 + MB^2 = (40\sqrt{3})^2 + 10^2 = 4900 \qquad \Longrightarrow \qquad AP = \boxed{70}.$$

2. Let $ABCD$ be an isosceles trapezoid with $AD = BC = 15$ such that the distance between its bases AB and CD is 7. Suppose further that the circles with diameters $\overline{AD}$ and $\overline{BC}$ are tangent to each other. What is the area of the trapezoid?

   Proposed by David Altizio

   Solution. Let T be the point of tangency of the two circles, and let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively. Then M, N, and T all lie on the same line, so

   $$MN = MT + TN = \frac{1}{2}AD + \frac{1}{2}BC = \frac{1}{2}\cdot 15 + \frac{1}{2}\cdot 15 = 15.$$

   Now recall that the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the distance between the bases of the trapezoid and b_1 and b_2 are said bases' lengths. But recall that MN is a midline of $ABCD$, meaning that its length is the average of the lengths of AB and CD. But this is precisely $\frac{1}{2}(b_1 + b_2)$! Therefore the desired area is $7 \cdot 15 = \boxed{105}$.

3. Let ABC be a triangle. The angle bisector of $\angle B$ intersects AC at point P, while the angle bisector of $\angle C$ intersects AB at a point Q. Suppose the area of $\triangle ABP$ is 27, the area of $\triangle ACQ$ is 32, and the area of $\triangle ABC$ is 72. The length of $\overline{BC}$ can be written in the form $m\sqrt{n}$ where m and n are positive integers with n as small as possible. What is $m + n$?

   Proposed by David Altizio

   Solution. For ease of typesetting let $[X]$ denote the area of region X. Note that $[ABP] = 27$ and $[ABC] = 72$ implies that $[BCP] = 72 - 27 = 45$, so by the Angle Bisector Theorem

   $$\frac{AB}{BC} = \frac{AP}{PC} = \frac{[ABP]}{[BPC]} = \frac{27}{45} = \frac{3}{5}.$$

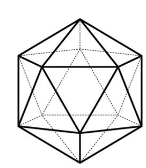
   Through a similar process one may obtain $\frac{AC}{BC} = \frac{4}{5}$. Therefore $\triangle ABC$ is a 3-4-5 right triangle with a right angle at A.

   Let $AB = 3x$, $AC = 4x$, and $BC = 5x$ for some positive real x. Then by the formula for area

   $$\frac{1}{2}(3x)(4x) = 72 \qquad \Longrightarrow \qquad x = \sqrt{12} = 2\sqrt{3}.$$

   Thus $BC = 5x = 10\sqrt{3}$ and the requested answer is $10 + 3 = \boxed{13}$.

4. Andrew the Antelope canters along the surface of a regular icosahedron, which has twenty equilateral triangle faces and edge length 4. (A three-dimensional image of an icosahedron is shown to the right.) If he wants to move from one vertex to the opposite vertex, the minimum distance he must travel can be expressed as $\sqrt{n}$ for some integer n. Compute n.

   Proposed by Patrick Lin

   Solution. Looking at the icosahedral net, it is clear that the desired length is equal to the hypotenuse of a right triangle with one leg equal to the height of a triangular face and the other leg equal to $\frac{5}{2}$ of the side length of a face. Hence Pythagorean theorem yields $10^2 + (2\sqrt{3})^2 = \boxed{112}$.

5. Let $\mathcal{P}$ be a parallelepiped with side lengths x, y, and z. Suppose that the four space diagonals of $\mathcal{P}$ have lengths 15, 17, 21, and 23. Compute $x^2 + y^2 + z^2$.

Proposed by David Altizio and Joshua Siktar

Solution. Recall the Parallelogram Law in two dimensions, which states that if x and y are elements of $\mathbb{R}^2$ then $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$. (This is true by the Law of Cosines.) I claim that this can be extended further. Indeed, for any three dimensional vectors x, y, and z in $\mathbb{R}^3$, the identity

$$4(|x|^2 + |y|^2 + |z|^2) = |x+y+z|^2 + |x+y-z|^2 + |x-y+z|^2 + |-x+y+z|^2$$

is true. To prove this, we use the two-dimensional version repeatedly. Note that 0, $x+y$, z, and $x+y+z$ form a parallelogram, which means that

$$2(|x+y|^2 + |z|^2) = |x+y+z|^2 + |x+y-z|^2.$$

Similarly, since 0, $x-y$, z, and $x-y+z$ form a parallelogram, we have

$$2(|x-y|^2 + |z|^2) = |x-y+z|^2 + |-x+y+z|^2.$$

Adding these together yields

$$2(|x+y|^2 + |x-y|^2) + 4|z|^2 = |x+y+z|^2 + |x+y-z|^2 + |x-y+z|^2 + |-x+y+z|^2$$

and using the parallelogram law on the LHS one last time yields the desired equality.

Returning back to the original problem, we have

$$4(x^2 + y^2 + z^2) = 15^2 + 17^2 + 21^2 + 23^2 = 2(16^2 + 1) + 2(22^2 + 1) = 2(16^2 + 22^2) + 4,$$

which means that $x^2 + y^2 + z^2 = 2(8^2 + 11^2) + 1 = \boxed{371}$.

Remark. This version of the parallelogram law can be extended to hold true in all dimensions. Formally,

$$2^n \sum_{i=1}^{n} |z_i|^2 = \sum_{(e_1,\ldots,e_n)\in\{+1,-1\}^n} \left| \sum_{i=1}^{n} e_i z_i \right|^2$$

for vectors $z_1, \ldots, z_n \in \mathbb{R}^n$.

6. In parallelogram $ABCD$, angles B and D are acute while angles A and C are obtuse. The perpendicular from C to AB and the perpendicular from A to BC intersect at a point P inside the parallelogram. If $PB = 700$ while $PD = 821$, what is AC?

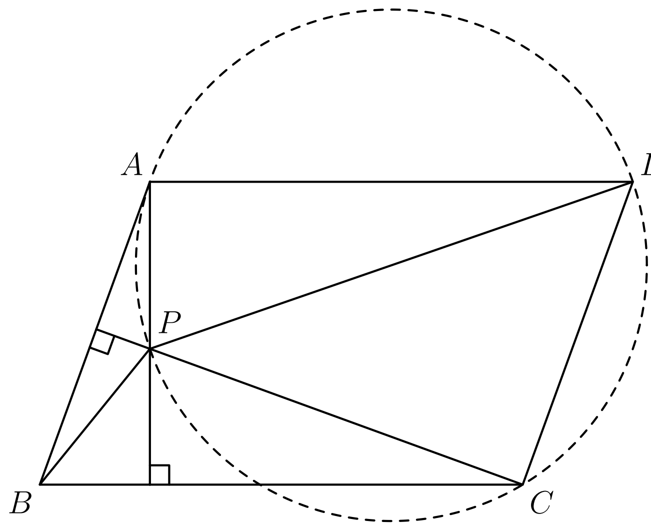
Proposed by David Altizio

Solution. First note that P is the orthocenter of $\triangle ABC$. Furthermore, note that from the perpendicularity $DA \perp AP$ and $DC \perp CP$, so quadrilateral $DAPC$ is cyclic. Furthermore, DP is a diameter of circle $(DAPC)$. This is the circumcircle of $\triangle DAC$, which is congruent to BCA. As a result, if R is the circumradius of $\triangle ABC$, then $PD = 2R$.

Now I claim that $PB = 2R\cos B$. To prove this, reflect P across AB to point P'. It is well-known that P' lies on the circumcircle of $\triangle ABC$, so in particular the circumradii of $\triangle APB$ and $\triangle ACB$ are equal. But then by Law of Sines

$$\frac{BP}{\sin\angle BAP} = \frac{BP}{\cos B} = 2R \implies BP = 2R\cos B$$

as desired. (An alternate way to see this is through the diagram itself: from right triangle trigonometry on triangles DAP and DCP it is not hard to see that $PA = 2R\cos A$ and $PC = 2R\cos C$, which by symmetry suggests $PB = 2R\cos B$.)



Finally, note that by Law of Sines again we have $AC = 2R\sin B$, so

$$AC^2 + BP^2 = (2R\sin B)^2 + (2R\cos B)^2 = (2R)^2(\sin^2 B + \cos^2 B) = PD^2.$$

Hence

$$AC^2 = PD^2 - PB^2 = 821^2 - 700^2 = (821 - 700)(821 + 700) = 11^2 \cdot 39^2$$

and so $AC = 11 \cdot 39 = \boxed{429}$.

7. Let ABC be a triangle with incenter I and incircle ω. It is given that there exist points X and Y on the circumference of ω such that $\angle BXC = \angle BYC = 90^\circ$. Suppose further that X, I, and Y are collinear. If $AB = 80$ and $AC = 97$, compute the length of BC.

   Proposed by David Altizio

   Solution. Let Ω be the circle with diameter $\overline{AC}$. Then X and Y are the intersection points of ω and Ω, so XY is the radical axis of ω and Ω. The condition that X, I, and Y are collinear implies that I lies on the radical axis of these two circles.

   Let M be the midpoint of $\overline{BC}$ and D the point of tangency of ω with BC. The power of I with respect to ω is r^2, while the power of I with respect to Ω is

$$MB^2 - MI^2 = \left(\frac{a}{2}\right)^2 - (ID^2 + DM^2) = \frac{a^2}{4} - \left(r^2 + \left(\frac{a}{2} - (s-b)\right)^2\right) = a(s-b) - r^2 - (s-b)^2.$$

   Setting these equal to each other yields

$$2r^2 = a(s-b) - (s-b)^2 = (s-b)(a+b-s) = (s-b)(s-c).$$

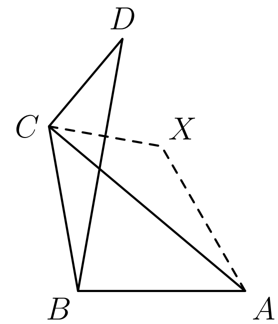
   Now recall that Heron's Formula states $(rs)^2 = K^2 = s(s-a)(s-b)(s-c)$. Plugging in our equality from above and cancelling like mad leads to

$$s = 2(s-a) \qquad \Longrightarrow \qquad a+b+c = 2(b+c-a) \qquad \Longrightarrow \qquad b+c = 3a.$$

   Hence $BC = \frac{AB+AC}{3} = \frac{80+97}{3} = \boxed{59}$.

8. Suppose $ABCD$ is a convex quadrilateral satisfying $AB = BC$, $AC = BD$, $\angle ABD = 80^\circ$, and $\angle CBD = 20^\circ$. What is $\angle BCD$ in degrees?

   Proposed by David Altizio



Solution. Construct a point X outside $\triangle ABC$ such that $\triangle BCD \cong \triangle AXC$. (This can be done from the fact that $AC = BD$.) Then from $AB = BC$ we know $\angle BAC = 40^\circ$, so $\angle BAX = 40^\circ + 20^\circ = 60^\circ$. Combining this with $AX = BC = BA$ gives that $\triangle ABX$ is equilateral.

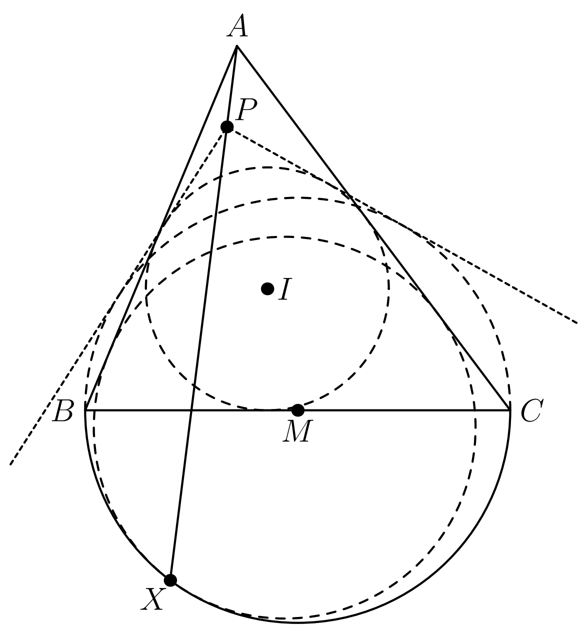
From here, note that $\angle CBX = \angle CBA - 60^\circ = 40^\circ$, and since $\triangle CBX$ is isosceles $\angle BXC = 70^\circ$. Thus

$$\angle BCD = \angle AXC = \angle BXC + \angle AXB = 70^\circ + 60^\circ = \boxed{130^\circ}.$$

9. Let $\triangle ABC$ be a triangle with $AB = 65$, $BC = 70$, and $CA = 75$. A semicircle Γ with diameter $\overline{BC}$ is erected outside the triangle. Suppose there exists a circle ω tangent to AB and AC and furthermore internally tangent to Γ at a point X. The length AX can be written in the form $m\sqrt{n}$ where m and n are positive integers with n not divisible by the square of any prime. Find $m + n$.

Proposed by David Altizio

Solution. Scale down by a factor of 5, so that $AB = 13$, $BC = 14$, and $CA = 15$. Let κ denote the incircle of $\triangle ABC$. The key is to recognize that by Monge's Theorem (or simply composite homotheties) AX passes through the exsimilicenter P of κ and Γ. Since both of these circles are fixed, P is also fixed. Thus it suffices to determine the location of P and use this to find the location of X.



Denote by I the incenter of $\triangle ABC$ and by M the midpoint of $\overline{BC}$. Furthermore, let I_0 and P_0 be the feet of the perpendiculars from I and P respectively to BC. We can easily compute that the radii of κ and Γ are 4 and 7 respectively, so by the definition of exsimilicenter, $\frac{PI}{PM} = \frac{4}{7}$. This in turn implies $\frac{P_0I_0}{P_0M} = \frac{4}{7}$. It is readily seen that $I_0M = 1$, so therefore $P_0M = \frac{7}{3}$. Similar reasoning yields $P_0P = \frac{28}{3}$.

Briefly turn to coordinates to make conceptualization easier. Set up a coordinate system where M is the origin and BC is the x-axis. Then P has coordinates $(-\frac{7}{3}, \frac{28}{3})$ and A has coordinates $(-2, 12)$, so the slope of line AP is 8.

Revert back to the Euclidean Plane. Let $D = AX \cap BC$, and let X_0 be the foot of the perpendicular from X to BC. Set X_0D to be t. Then $XX_0 = 8t$. Furthermore, DM can be easily computed to be $\frac{3}{2} + 2 = \frac{7}{2}$ by the definition of slope. Thus by Pythagorean Theorem on $\triangle MX_0X$,

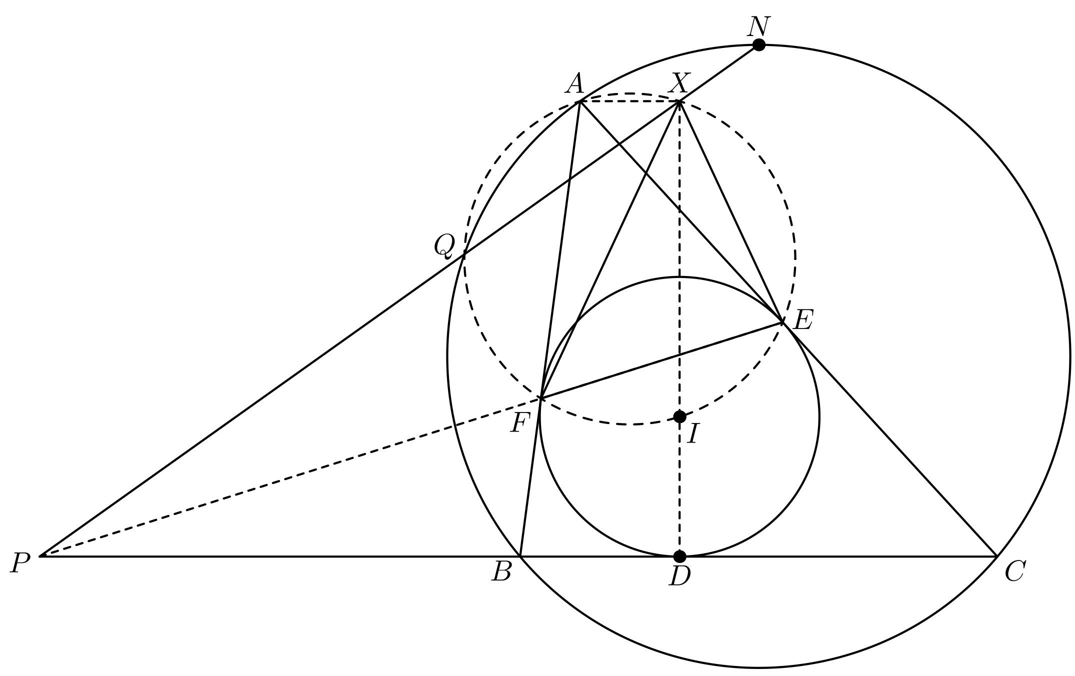
$$\left(t + \frac{7}{2}\right)^2 + (8t)^2 = 7^2 \qquad \Longrightarrow \qquad t = \frac{7}{10}.$$

A few applications of the Pythagorean Theorem yield $AX = \frac{11\sqrt{65}}{5}$. Scaling back up by a factor of 5 yields $m = 11$, $n = 65$, and $m + n = \boxed{076}$.

10. Let $\triangle ABC$ be a triangle with circumcircle Ω and let N be the midpoint of the major arc $\widehat{BC}$. The incircle ω of $\triangle ABC$ is tangent to AC and AB at points E and F respectively. Suppose point X is placed on the same side of EF as A such that $\triangle XEF \sim \triangle ABC$. Let NX intersect BC at a point P. If $AB = 15$, $BC = 16$, and $CA = 17$, then compute $\frac{PX}{XN}$.

Proposed by David Altizio

Solution. We solve for general a, b, and c. We start off by proceeding through a series of lemmas.



LEMMA 1: $AX \parallel BC$.

Proof. Let I be the incenter of $\triangle ABC$. Note that since $\angle EXF = \angle EAF$, X lies on the circumcircle of $\triangle AEF$. Now remark that since

$$\angle FID + \angle FIX = 180^\circ - \angle B + \angle XEF = 180^\circ - \angle B + \angle B = 180^\circ,$$

we have D, I and X collinear, i.e. $XI \perp BC$. Furthermore, A and I are antipodal with respect to (AEF), so $\angle AXI = 90^\circ$. Hence $AX \parallel BC$ as desired. $\square$

LEMMA 2: Denote by Q the second intersection point of Ω with (AEF). Then Q lies on $\overline{PN}$.

Proof. Extend AX to hit Ω again at A'. Then $AA'CB$ is an isosceles trapezoid. Furthermore, N is the midpoint of $\widehat{AA'}$, so $\angle AQN = \angle A'QN$.

Now consider the spiral similarity sending $\triangle XEF$ to $\triangle A'BC$. This spiral similarity is centered at Q (a well-known fact - provable by angle chasing). Since this spiral similarity sends A to N (both are midpoints of their respective arcs), we have $\triangle QAX \sim \triangle QNA'$, i.e. $\angle AQX = \angle NQA'$. Hence N, X, and Q are collinear, leading to the desired. $\square$

LEMMA 3: EF passes through P.

Proof. Note that by simple angle chasing

$$\angle XQF = 180^\circ - \angle XEF = 180^\circ - \angle ABC = \angle PBF.$$

This implies that quadrilateral $PQFB$ is cyclic, so $\angle PBQ = \angle PFQ$. But since Q is the center of spiral similarity sending EF to BC, we also have $\angle QFE = \angle QBC$. Hence since P, B, and C are collinear we must also have P, E, and F collinear. $\square$

Now we compute. Remark that by Power of a Point $PQ \cdot PX = PF \cdot PE = PD^2$ and $PQ \cdot PN = PB \cdot PC$, so

$$\frac{XN}{PX} = \frac{PN}{PX} - 1 = \frac{PN \cdot PQ}{PX \cdot PQ} - 1 = \frac{PB \cdot PC}{PD^2} - 1.$$

To compute PB, remark that by either Menelaus or harmonic divisions we may obtain $\frac{PB}{PC} = \frac{DB}{DC}$. Since $BD = s - b$ and $CD = s - c$, it is easy to find that $PB = \frac{a(s-b)}{b-c}$. This means that $PC = \frac{a(s-c)}{b-c}$ and

$$PD = \frac{a(s-b)}{b-c} + (s-b) = (s-b)\left(\frac{a}{b-c} + 1\right) = (s-b)\left(\frac{a+b-c}{b-c}\right) = \frac{2(s-c)(s-b)}{b-c}.$$

As a result,

$$\frac{PB \cdot PC}{PD^2} = \frac{a^2(s-b)(s-c)/(b-c)^2}{(2(s-b)(s-c)/(b-c))^2} = \frac{a^2}{4(s-b)(s-c)}.$$

Hence

$$\frac{XN}{PX} = \frac{a^2}{4(s-b)(s-c)} - 1 = \frac{16^2}{4(24-15)(24-17)} - 1 = \frac{8^2}{9 \cdot 7} - 1 = \frac{1}{63},$$

so $\frac{PX}{XN} = \boxed{63}$.